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ON ENTIRE FUNCTIONS DEFINED BY A DIRICHLET SERIES.*

By S. Mandelbrojt and J. J. Gergen.†

1. *Introduction.* It is a well-known fact that there corresponds to every integral transcendental function $f(z)$ at least one ray L, issuing from the origin, such that in every angle, with vertex at the origin, containing L, $f(z)$ assumes each value, with possibly one exception.‡ Moreover, that there are certain analogies between these half-lines, which are ordinarily called the "lines of Julia," or "lines J," of $f(z)$, and the singularities of a non-integral function has been pointed out by Bloch § and concretely demonstrated by several authors. Biernacki ¶ has proved that if $f(z)$ is an entire function of non-zero order, at least one of its lines J is also a line J with regard to all the derivatives and integrals of $f(z)$. Gontcharoff ‖ has shown that if E is an arbitrary closed set of half-lines issuing from the origin, then there exist entire functions of order $\rho \leq 1/2$ whose lines J coincide with E, and entire functions of order $\rho > 1/2$ whose lines J coincide with the subset of E situated in the angle with vertex at the origin and opening equal to the larger of the two numbers π/ρ, $2\pi - \pi/\rho$. Finally, Pólya has found analogues to several of the theorems on the singularities of a function defined by a lacunary Taylor series. In particular, he established recently the second of the two related propositions:

* A résumé of the theorems contained in this paper was published in a note, "Sur les fonctions définies par une série de Dirichlet," in *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 189 (1929), pp. 1057-1059.

† National Research Fellow.

‡ This proposition is due to Julia. A proof may be found in Montel, *Leçons sur les familles normales de fonctions analytiques*, Paris (1927), pp. 81-85. Hereafter we shall refer to this book by the letter *M*.

§ Bloch, "Les fonctions holomorphes et méromorphes dans le cercle-unité," *Mémorial des Sciences Mathématiques*, Paris (1926), p. 15.

¶ Biernacki, "Sur les droites de Julia des fonctions entières," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 186 (1928), pp. 1260-1262 and pp. 1410-1412. See also Biernacki, "Sur la theorie des fonctions entières," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A (1929), p. 529.

‖ Gontcharoff, "Sur les fonctions entières et les droits de Julia," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 185 (1927), pp. 1100-1102.

THEOREM A.* *If the maximum density* † *of the non-vanishing coefficients of a power series*

$$\psi(z)=\sum_{n=0}^{\infty} b_{\lambda_n} z^{\lambda_n},$$

with unit radius of convergence, is equal to Δ, *then, on every arc of the unit circle of length* $2\pi\Delta$, $\psi(z)$ *has at least one singular point.*

THEOREM B.‡ *If the maximum density of the non-vanishing coefficients of an entire function*

$$\psi(z)=\sum_{n=0}^{\infty} b_{\lambda_n} z^{\lambda_n},$$

of infinite order, is equal to Δ, *then, in every angle with vertex at the origin and opening* $2\pi\Delta$, $\psi(z)$ *has a line* J.

Theorem A has, of course, been generalized to Dirichlet series,§

$$(1) \qquad f(s)=\sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \qquad (s=\sigma+it)$$

$$0=\lambda_0<\lambda_1<\lambda_2 \cdots; \quad \lim_{n=\infty} \lambda_n=\infty; \quad a_{\lambda_n}\neq 0, \quad n>0.$$

In particular, it has been shown ¶ that, if

$$\lim_{n=\infty} (\lambda_{n+1}-\lambda_n) \geq G>0$$

---

* This proposition is Pólya's generalization of a theorem by Fabry. Fabry's original theorem may be found in his paper "Sur les séries de Taylor qui ont une infinité de points singuliers," *Acta Mathematica*, Vol. 22 (1898), on p. 86. For references to Pólya's generalization and related theorems, see Pólya's paper "Untersuchungen über Lücken und Singularitäten von Potenzreihen," *Mathematische Zeitschrift*, Vol. 29 (1929), p. 626.

† That is, if

$$\lim_{\xi=1-0} \overline{\lim_{r=\infty}} \left[\{N(r)-N(r\xi)\}/(r-r\xi)\right]=\Delta,$$

where $N(r)$ is the number of λ_n's not greater than r. Cf. Pólya, *loc. cit.*, p. 559.

‡ Cf. Pólya, *loc. cit.*, p. 626.

§ The condition that a_{λ_n} be different from zero for n different from zero is imposed for purposes of exposition. It involves no loss of generality except in Theorem VI.

¶ This is a particular case of a theorem of Pólya. For references to this and related theorems see Valiron, "Théorie générale des séries de Dirichlet," *Mémorial des Sciences Mathématiques*, Paris (1926), pp. 21-25. V. Bernstein has recently obtained some results in the same order of ideas. "Sur les points singuliers des fonctions representées par des séries de Dirichlet," *Comptes Rendus des Séances de l'Académie des Sciences*, Vol. 188 (1929), p. 539.

and the axis c of convergence of (1) is finite, then $f(s)$ has at least one singular point on every segment of c of length $2\pi/G$.

Theorem B has as yet, however, never been generalized to these series; and it is in part the purpose of this paper to consider this problem. With this in mind the lines of interest are clearly horizontal lines parallel to the axis of reals, and the regions some horizontal strips. Rather, however, than confine our attention to regions in which the function in question assumes each value, save possibly one, only once, we shall consider those in which it assumes each value, with the possible exception, infinitely many times. Since the chief weapon in the analysis is the theory of normal families this makes no appreciable difference in the proofs. Our propositions are concerned with lines $\bar{J}$ and J_1 defined as follows:

Definition. A line L, parallel to the axis of reals, is a line $\bar{J}$ of the function $f(s)$ if, in every horizontal strip containing L, $f(s)$ assumes each value, with possibly one exception, infinitely many times.

Definition. A ray L, issuing from the origin, is a half-line J_1 of $f(s)$ if, in every angle, with vertex at the origin, containing L, $f(s)$ assumes each value, with possibly one exception, infinitely many times.

Theorems I to IV of Section 2 are on the lines $\bar{J}$ of the series (1). In Theorem II we obtain an analogue of the above mentioned theorem on the singularities of (1). In Theorem III we suppose that the λ_n's are integers and obtain a generalization of Pólya's result *B*, weakening the condition on the order of $f(s)$. We employ in the proof a lemma of Pólya. The theorems of Section 3 are on the lines J_1 of the series (1).

2.1. *The Lines $\bar{J}$.* THEOREM I. *Suppose that the series* (1) *is everywhere* (*absolutely* *) *convergent and that* $\varliminf_{n=\infty} (\lambda_n/n) \geq G > 0$. *Then, if T, $|t-\bar{t}| \leq \pi/G$, be any horizontal strip of width $2\pi/G$, either $f(s)$ has a line $\bar{J}$ in T, or else*

$$\lim_{m=\infty} |f(s_m)e^{us_m}| = \infty$$

for every u and every sequence of points $\{s_m = \sigma_m + it_m\}$ such that

$$(2) \qquad \lim_{m=\infty} \sigma_m = -\infty, \qquad \overline{\lim_{m=\infty}} |t_m - \bar{t}| \leq \pi/G.$$

The proof depends partly upon certain results found in the theory of normal families and partly upon the following lemma:

* It is known that for series of this type the axis of simple convergence coincides with the axis of absolute convergence. Cf. Valiron, *loc. cit.*, p. 3.

LEMMA. *If* $0 < \lambda_1 < \lambda_2 \cdots$ *and* $\varliminf_{n=\infty} (\lambda_n/n) \geq G > 0$, *then*

$$g_j(z) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} z^{2m+1} = z \prod_{n=1}^{\infty}{}' (1 - z^2/\lambda_n{}^2) \qquad (n \neq j),\ j = 1, 2, \cdots$$

is entire, and for every positive number ϵ

$$(2m+1)!\,|\,C^{(j)}{}_{2m+1}\,| \leq A_1(\epsilon)\{\pi(1+\epsilon)/G\}^{2m+1}, \tag{3}$$

where $A_1(\epsilon)$ *is independent of* m *and* j.

That $g_j(z)$ is entire is well known. Let then n_0 be chosen so large that

$$\lambda_n \geq nG/(1+\epsilon)$$

for $n \geq n_0$, and let

$$A_2(\epsilon) = \{n_0(\lambda_1 + 1)(G+1)/\lambda_1\}^{n_0}.$$

We have for any product $\lambda = \lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m}$ of distinct λ_n's

$$A_2\lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m} \geq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m. \tag{4}$$

In fact, if each factor λ_{k_j} is less than λ_{n_0},

$$\begin{aligned} A_2\lambda &\geq A_2k_1k_2 \cdots k_m\lambda_1{}^m/n_0{}^m \\ &\geq A_2k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m\{\lambda_1/(Gn_0)\}^m \\ &\geq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m, \end{aligned} \tag{5}$$

whereas, if no factor is less than λ_{n_0},

$$\lambda \geq k_1k_2 \cdots k_m\{G/(1+\epsilon)\}^m, \tag{6}$$

and (5) and (6) together show that (4) holds in general.

Now for $m \geq 1$,

$$\begin{aligned} |\,C^{(j)}{}_{2m+1}\,| &= \sum_{k_1=1}^{\infty}{}' \sum_{k_2=k_1+1}^{\infty}{}' \cdots \sum_{k_m=k_{m-1}+1}^{\infty}{}' (\lambda_{k_1}\lambda_{k_2} \cdots \lambda_{k_m})^{-2} \qquad (k_p \neq j) \\ &\leq A_2{}^2\{(1+\epsilon)/G\}^{2m} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_m=k_{m-1}+1}^{\infty} (k_1k_2 \cdots k_m)^{-2} \\ &= A_2{}^2\{\pi(1+\epsilon)/G\}^{2m}/(2m+1)!, \end{aligned}$$

the last relation being deduced from the identities

$$\pi ix \prod_1^{\infty} (1 + x^2/n^2) = \sin \pi ix = i \sum_{m=0}^{\infty} (\pi x)^{2m+1}/(2m+1)!.$$

Accordingly, since

$$|\,C_1{}^{(j)}\,| = 1 \leq A_2{}^2,$$

(3) holds with $A_1 = GA_2{}^2$.

2.2. Returning now to the proof of Theorem I, we consider the functions *

$$\phi_j(s) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} f^{2m+1}(s), \tag{7}$$

where the $C^{(j)}{}_{2m+1}$ are defined as above. From the inequalities

$$r^{2m+1} \max_{|s| \leq r} | f^{2m+1}(s) | \leq 2(2m+1)! \max_{|s| \leq 2r} | f(s) | \qquad (0 < r), \tag{8}$$

we deduce that the series (7) converges uniformly in every closed region, and hence that $\phi_j(s)$ is entire. Moreover, denoting by $F(s)$ the series

$$F(s) = \sum_{n=0}^{\infty} | a_n | e^{-\lambda_n s},$$

we have for $\sigma \geq 0$

$$\begin{aligned} r^{2m+1} \sum_{n=1}^{\infty} | a_n(-\lambda_n)^{2m+1} e^{-\lambda_n s} | &= r^{2m+1} \sum_{n=1}^{\infty} | a_n | \lambda_n{}^{2m+1} e^{-\lambda_n \sigma} \\ &\leq r^{2m+1} | F^{(2m+1)}(0) | \\ &\leq (2m+1)!\, F(-r), \end{aligned}$$

so that in the series (7) it is permissible, when $\sigma \geq 0$, to replace $f^{2m+1}(s)$ by its expansion $\sum_1^{\infty} a_n(-\lambda_n)^{2m+1} e^{-\lambda_n s}$ and to change the order of summation. If this is done, we get

$$\phi_j(s) = -\sum_{n=0}^{\infty} a_n g_j(\lambda_n) e^{-\lambda_n s} = -a_j g_j(\lambda_j) e^{-\lambda_j s}, \tag{9}$$

which, being valid for $\sigma \geq 0$, is valid throughout the plane.

From (9) it now follows that for every positive ϵ

$$| a_j g_j(\lambda_j) e^{-\lambda_j s} | \leq A_3(\epsilon) \underset{|z-s| = r_\epsilon}{\text{maximum}} | f(z) | = A_3 N(s), \tag{10}$$

where $r_\epsilon = (\pi + \epsilon)/G$, and A_3 is independent of j and s. In fact,

$$| f^{2m+1}(s) | \leq (2m+1)! \{G/(\pi+\epsilon)\}^{2m+1} N(s),$$

and so

$$\begin{aligned} | a_j g_j(\lambda_j) e^{-\lambda_j s} | &= | \phi_j(s) | \\ &\leq \sum_{m=0}^{\infty} | C^{(j)}{}_{2m+1} | \, | f^{2m+1}(s) | \\ &\leq A_1(\kappa) N(s) \sum_{m=0}^{\infty} \{2\pi/(2\pi+\epsilon)\}^{2m+1} \\ &= A_3(\epsilon) N(s), \end{aligned}$$

where $\kappa = \epsilon/(2\pi + \epsilon)$.

---

* Functions of this type have been used in similar circumstances by several authors. For references see Valiron, *loc. cit.*, pp. 21-25.

With the aid of (10) we prove that, if for some u the function $|f(s)s^{us}|$ is bounded on a sequence of points $\{s_m = \sigma_m + it_m\}$ satisfying (2), then, no matter how small the positive number ϵ, the family of functions

$$f_n(s) = f(s - 2n\pi/G) \qquad (n = 0, 1, 2, \cdots) \tag{11}$$

is not normal * in the square R_3

$$|\sigma| < (\pi + 3\epsilon)/G, \quad |t - \bar{t}| < (\pi + 3\epsilon)/G.$$

This, of course, is sufficient to establish the theorem. For under these circumstances there is at least one point in R_3 at which the family is not normal,† and thus, since ϵ is arbitrary, there is at least one point $s' = \sigma' + it'$ in the square R_0

$$|\sigma| \leq \pi/G, \quad |t - \bar{t}| \leq \pi/G,$$

at which it is not normal. Accordingly,‡ in every horizontal strip containing the line $t = t'$, $f(s)$ assumes each value, with one possible exception, infinitely many times. Hence the line $t = t'$ is a line $\bar{\bar{J}}$.

We observe firstly that, if $|f(s)e^{us}|$ is bounded on some sequence of points satisfying (2), then $|f(s)|$ is bounded on a set of points $\{s_m' = \sigma_m' + it_m'\}$ such that

$$\lim_{m=\infty} \sigma_m' = -\infty, \quad |t - \bar{t}| \leq (\pi + 2\epsilon)/G. \tag{12}$$

In fact, if the contrary were true, we should have for all n sufficiently large, say $n \geq n_1$,

$$|f_n(s)| > 1$$

for all s in the square R_2

$$|\sigma| \leq (\pi + 2\epsilon)/G, \quad |t - \bar{t}| \leq (\pi + 2\epsilon)/G\,;$$

and thus, by a theorem of Mandelbrojt,§ for all $n \geq n_1$ and s and z in the square R_1

$$|\sigma| \leq (\pi + \epsilon)/G, \quad |t - \bar{t}| \leq (\pi + \epsilon)/G,$$

it would follow that

$$\log |f_n(s)| \leq \alpha_{R_1} \log |f_n(z)|, \tag{13}$$

---

* For the definition of normality see *M*, p. 32.

† *M*, p. 34.

‡ *M*, p. 61. Evidently the normality or non-normality of a sequence of functions is independent of any finite group of functions.

§ Mandelbrojt, "Sur les suites de fonctions holomorphes," *Journal de Mathématiques* (*Liouville*), Vol. 18 (1929), p. 176.

where α_{R_1} is independent of s, z and n. But, on the other hand, because of (10) and our assumption as to the boundedness of $|f(s)e^{us}|$, we can choose a subsequence $\{f_{n_k}(s)\}$ from the sequence (11), and two sets of points $\{z_k\}$, $\{z_k'\}$, all contained in R_1, such that

$$\lim_{k=\infty} (n_k^{-1} \log |f_{n_k}(z_k)|) = \infty,$$
$$0 < \log |f_{n_k}(z_k')| < A_4 n_k,$$

where A_4 is independent of k. We thus have

$$\lim_{k=\infty} \{(\log |f_{n_k}(z_k')|)^{-1} \log |f_{n_k}(z_k)|\} = \infty,$$

which evidently contradicts (13). Hence the boundedness of $|f(s)e^{us}|$ implies that of $|f(s)|$ itself.

To complete the proof we have then only to show that, if $|f(s)|$ is bounded on a set of points satisfying (12), the family (11) is not normal in R_3. For this we observe that under these conditions a subsequence $\{f_{n_k}(s)\}$ can be chosen from the sequence (11), together with two sets of points $\{z_k\}$, $\{z_k'\}$, all contained in R_2, such that

$$\lim_{k=\infty} |f_{n_k}(z_k)| = \infty, \quad |f_{n_k}(z_k')| \leq A_5,$$

where A_5 is independent of k. It is then evident that the sequence $\{f_{n_k}(s)\}$ cannot converge uniformly to a holomorphic function in R_2, and that the sequence $\{|f_{n_k}(s)|\}$ cannot become infinite uniformly there. Thus the family $\{f_n\}$ is not normal in R_3; and the theorem is proved.

2. 3. In our next theorem we employ the notion of order, as defined by Ritt,* of an entire function $f(s)$ represented by an everywhere absolutely convergent Dirichlet series. *This order, which we shall call the R-order of* $f(s)$, *is, by definition, the* †

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1} \log_2 M_f(\sigma)\},$$

where $M_f(\sigma)$ *is the least upper bound of* $|f(s)|$ *on the vertical line with abscissa* σ. It is to be noted, firstly, that the R-order is a natural generalization of ordinary order, inasmuch as the R-order of the entire function

* Ritt, "On Certain Points in the Theory of Dirichlet Series," *American Journal of Mathematics*, Vol. 50 (1928), p. 77.

† By definition

$$\log_2 A = \begin{cases} \log \log A, & A > 1, \\ 0 \quad , & A \leq 1. \end{cases}$$

$$\psi(s) = \sum_{n=0}^{\infty} a_n e^{-ns}$$

is equivalent to the ordinary order of $\psi_1(z) = \psi(-\log z)$, and secondly that, as in the case of ordinary order, we have the fundamental theorem,* that the R-order of

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

is ρ when and only when

$$\overline{\lim_{n=\infty}} \, \{(\lambda_n \log \lambda_n)^{-1} \log | a_n | \} = -1/\rho, \tag{14}$$

this conclusion holding if the series is everywhere absolutely convergent and

$$\underline{\lim_{n=\infty}} \, (\lambda_n/\log n) > 0.$$

This formula (14) is important for us. By means of it we are able to eliminate the second of the two possibilities of Theorem I. We prove

2.4. THEOREM II. *If the series* (1) *is everywhere (absolutely) convergent and* $\underline{\lim}_{n=\infty} (\lambda_{n+1} - \lambda_n) \geq G > 0$, *and if the R-order of $f(s)$ is* $\geq \rho > 0$, *then $f(s)$ has a line $\bar{J}$ in every horizontal strip of width $2\pi/\alpha$, where α is the smaller of the two numbers 2ρ and G.*

The proof rests on the two following lemmas.

LEMMA 1. *If*

$$\mu_{n+1} - \mu_n \geq G/2 > 0, \quad nG \leq \mu_n < (n+1)G \qquad (n = 1, 2, \cdots),$$

then, for $j \geq 2$,

$$G^3 \leq 16\pi\mu_j^2 \, | g_j(\mu_j) | = 16\pi\mu_j^3 \prod_1^{\infty}{}' \, | 1 - \mu_j^2/\mu_n^2 | \qquad (n \neq j). \tag{15}$$

We have, writing $\mu_j = (j + \eta)G = \mu G$,

$$\begin{aligned} | g_j(\mu_j) | &= \mu_j(\mu_j^2/\mu^2_{j-1} - 1)(1 - \mu_j^2/\mu^2_{j+1}) \prod_{n=1}^{j-2} (\mu_j^2/\mu_n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu_j^2/\mu_n^2) \\ &\geq jG/(j+2)^2 \prod_{n=2}^{j-1} (\mu^2/n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu^2/n^2) \\ &\geq jG^3/\{\mu_j^2(j+2)^2\} \prod_{n=1}^{j-1} (\mu^2/n^2 - 1) \prod_{n=j+2}^{\infty} (1 - \mu^2/n^2). \end{aligned}$$

Hence, if $0 < \eta < 1$,

$$| \mu_j^2 g_j(\mu_j) | \geq j^3 G^3 \, | \sin \pi\eta | \, / \{4\pi(j+2)^3 \eta(1-\eta)\}.$$

---

* Ritt, *loc. cit.*, p. 78.

Now

$$| \sin \pi\eta | \geq 2\eta(1-\eta),$$

so that

$$| \mu_j^2 g_j(\mu_j) | \geq j^3G^3/\{2\pi(j+2)^3\} > G^3/(16\pi)$$

and (15) holds for $0 < \eta < 1$. But $g_j(z)$ is a continuous function of z; hence (15) also holds for $\eta = 0$.

LEMMA 2. *Let ϵ be an arbitrary positive number, and T', $|t-\bar{t}| \leq (\pi+2\epsilon)/G$, an arbitrary strip of width $2(\pi+2\epsilon)/G$. Then under the hypotheses of Theorem II,*

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1} \log_2 M_f(\sigma, T')\} \geq \rho,$$

where $M_f(\sigma, T')$ is the maximum of $|f(s)|$ on the segment of the vertical line of abscissa σ contained in T'.

We first choose a positive integer n_0 so large that

$$\lambda_{n+1} - \lambda_n \geq 2\pi G/(2\pi+\epsilon) = G_1$$

when $n \geq n_0 - 1$, and then select a sequence of numbers $\mu_1, \mu_2, \cdots$ containing the sequence $\lambda_{n_0}, \lambda_{n_0+1}, \cdots$ and such that

$$nG_1 \leq \mu_n < (n+1)G_1, \quad \mu_{n+1} - \mu_n \geq G_1/2 \tag{16}$$

for $n = 1, 2, \cdots$.

Now, writing

$$h(s) = \sum_{n=n_0}^{\infty} a_n e^{-\lambda_n s}$$

$$g_j(z) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} z^{2m+1} = z \prod_{n=1}^{\infty}{}' (1 - z^2/\mu_n^2) \qquad (n \neq j),$$

we have, as before,

$$\phi_j(s) = \sum_{m=0}^{\infty} C^{(j)}{}_{2m+1} h^{2m+1}(s) = - \sum_{n=n_0}^{\infty} a_n g_j(\lambda_n) e^{-\lambda_n s};$$

and, in particular, if j_k denotes an integer such that $\mu_{j_k} = \lambda_{n_0+k}$, this reduces to

$$\phi_{j_k}(s) = -a_{n_0+k} g_{j_k}(\mu_{j_k}) e^{\lambda_{n_0+k} s}.$$

Hence, by Lemma 1,

$$0 < A_5(\epsilon) \, | a_{n_0+k} e^{\lambda_{n_0+k} s} | \leq \lambda_{n_0+k}^2 \, | \phi_{j_k}(s) | ,$$

where A_5 is independent of s and k. But, by the lemma of Section 2.1,

$$| \phi_{j_k}(s) | \leq N_h(s) \sum_{m=0}^{\infty} (2m+1)! \, C^{(j)}{}_{2m+1} \{G/(\pi+2\epsilon)\}^{2m+1}$$

$$\leq A_1(\kappa)N_h(s) \sum_{m=0}^{\infty} [2G\pi(\pi+\epsilon)/\{G_1(\pi+2\epsilon)(2\pi+\epsilon)\}]^{2m+1}$$
$$= A_1N_h(s) \sum_{m=0}^{\infty} \{(\pi+\epsilon)/(\pi+2\epsilon)\}^{2m+1}$$
$$= A_6(\epsilon)N_h(s),$$

where $N_h(s)$ is the maximum of $|h(z)|$ on the circle $|s-z|=(\pi+2\epsilon)/G$, $\kappa=\epsilon/(2\pi+\epsilon)$ and A_6 is independent of s and j. Accordingly, we have for $n \geq n_0$ and all s

$$0 < A_7(\epsilon)\,|a_n e^{-\lambda_n s}| \leq \lambda_n{}^2 N_h(s), \tag{17}$$

where A_7 is independent of s and j.

By employing this inequality, it is not difficult to show that

$$\overline{\lim_{\sigma=-\infty}}\ \{(-\sigma)^{-1}\log_2 M_h(\sigma, T')\} \geq \rho, \tag{18}$$

where $M_h(\sigma, T')$ has the same meaning with regard to h as $M_f(\sigma, T')$ has with regard to f. Suppose that δ is arbitrarily small but positive. Let

$$\sigma_n = -\{(1-\delta)\rho\}^{-1}\log\lambda_n.$$

Let s_n be the point $\sigma_n + i\bar{t}$ and let $s_n' = \sigma_n' + it_n'$ be the point on the circle $|s-s_n| = (\pi+2\epsilon)/G$ at which $|h(s)|$ is a maximum. Writing

$$P = \exp[-\rho(1-\delta)(\pi+2\epsilon)/G]$$

we have, from (17),

$$\begin{aligned} \exp[\rho(1-\delta)\sigma_n'] \log M_h(\sigma_n', T') &\geq \rho\exp[\sigma_n\rho(1-\delta)] \log N_h(s_n) \\ &\geq P\lambda_n^{-1}[\log(A_7\lambda_n^{-2}) + \log|a_n| + \{\rho(1-\delta)\}^{-1}\lambda_n\log\lambda_n] \\ &\geq P\log\lambda_n\,[(\lambda_n\log\lambda_n)^{-1}\log(A_7\lambda_n^{-2}) \\ &\quad + (\lambda_n\log\lambda_n)^{-1}|a_n| + \{\rho(1-\delta)\}^{-1}]. \end{aligned} \tag{19}$$

Thus, from (14),

$$\overline{\lim_{n=-\infty}}\ \{e^{\sigma_n'\rho(1-\delta)}\log M_h(\sigma_n', T')\} = \infty;$$

and this, of course, proves (18).

To conclude the proof of the lemma it is necessary now only to observe that

$$M_f(\sigma, T') \geq M_h(\sigma, T') - Be^{-\lambda_{n_0}\sigma},$$

where B is independent of σ, for this implies that

$$\overline{\lim_{\sigma=-\infty}}\ \{(-\sigma)^{-1}\log_2 M_f(\sigma, T')\} = \overline{\lim_{\sigma=-\infty}}\ \{(-\sigma)^{-1}\log_2 M_h(\sigma, T')\} \geq \rho.$$

2.5. As a consequence of Lemma 2, Theorem I, and a theorem due to Valiron, the proof of II is almost immediate. Let us suppose that T, $|t-\bar{t}| \leq \pi/\alpha$ is an arbitrary horizontal strip of the plane. Then if ϵ is

arbitrary but positive, $\alpha_\epsilon = \alpha(1-\epsilon)$, and T_ϵ is the horizontal strip $|t - \bar{t}| \leq 2\pi/\alpha_\epsilon$, we have

$$\overline{\lim_{\sigma=-\infty}} \{(-\sigma)^{-1} \log_2 M_f(\sigma, T_\epsilon)\} > \alpha_\epsilon/2,$$

where $M_f(\sigma, T_\epsilon)$ has the same meaning with regard to T_ϵ as $M_f(\sigma, T')$ has with regard to T'. In fact,

$$2\pi/\alpha_\epsilon \geq 2\pi/\{G(1-\epsilon)\} > 2(\pi + 2\epsilon)/G,$$

whereas

$$\rho \geq \alpha_\epsilon/\{2(1-\epsilon)\} > \alpha_\epsilon/2.$$

Now, according to Valiron,* if $E(z)$ is holomorphic in an angle S' containing an angle S, the latter having its vertex at the origin and an opening $2\pi/\gamma$, and if

$$\overline{\lim_{r=0}} \{(-\log r)^{-1} \log_2 M_E'(r, S)\} > \gamma/2,$$

where $M_E'(r, S)$ is the maximum of $E(z)$ on the arc of the circle $|z| = r$ contained in S, then $E(z)$ assumes each value, save possibly one, on an infinite sequence of points in S' admitting the origin as a limit point. By making then the transformation $z = e^s$, we find that $|f(s)|$ is bounded on a sequence of points $\{s_m = \sigma_m + it_m\}$ such that

$$\overline{\lim_{m=\infty}} \sigma_m = -\infty, \quad |t_m - \bar{t}| \leq \pi(1+\epsilon)/\alpha_\epsilon.$$

This together with Theorem I is sufficient to establish II, for

$$\underline{\lim_{n=\infty}} (\lambda_n/n) \geq G > \alpha_\epsilon/(1+\epsilon),$$

and therefore in the strip

$$|t - \bar{t}| \leq 2\pi(1+\epsilon)/\alpha_\epsilon$$

$f(s)$ has a line $\bar{J}$. But ϵ is arbitrarily small and $\alpha_\epsilon - \alpha$ tends to zero with ϵ. Hence there is a line $\bar{J}$ in T.

2. 6. When the λ_n's are integers the conditions on the gaps in Theorem II may be made more general. Pólya † has shown that, if an entire function $\psi(z)$ of order ρ is represented by a series

$$\psi(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n},$$

in which the maximum density of the sequence $\{\lambda_n\}$ is Δ, then $\psi(z)$ is

* Valiron, "Fonctions entières et fonctions méromorphes d'une variable," *Mémorial des Sciences Mathématiques*, Paris (1925), p. 15.

† Cf. Pólya, *loc. cit.*, p. 622.

effectively of the order ρ in every sector S with vertex at the origin and opening $2\pi\Delta$. That is to say,

$$\overline{\lim_{r=\infty}} \{(\log r)^{-1} \log_2 M\psi'(r, S)\} = \rho,$$

where $M\psi'(r, S)$ is the maximum of $|\psi|$ in the arc of the circle $|z| = r$ contained in S. By observing that* $\underline{\lim}_{n=\infty} (\lambda_n/n) \geq 1/\Delta$ and employing this result rather than Lemma 2 in Section 2. 5, we deduce

THEOREM III. *Suppose that in the series* (1) *the* λ_n's *are integers, that their maximum density is* Δ, *and that* $f(s)$ *is an entire function of R-order* $\geq \rho > 0$. *Then* $f(s)$ *has a line* $\bar{J}$ *in every horizontal strip of width* $2\pi/\alpha$, *where* α *is the smaller of the two numbers* 2ρ *and* $1/\Delta$.

2. 7. We conclude our theorems on the lines $\bar{J}$ with the following:

THEOREM IV. *If the series* (1) *is everywhere* (*absolutely*) *convergent and* $\underline{\lim}_{n=\infty} (\lambda_{n+1} - \lambda_n) = \infty$, *and if* $f(s)$ *is of unbounded R-order, then every horizontal line is a line* $\bar{J}$ *of* $f(s)$.

This is evidently an immediate corollary of Theorem II.

3. 1. *The Lines* J_1. Turning now to the lines J_1 of a function represented by a Dirichlet series, we prove

THEOREM V. *Suppose that the series* (1) *is everywhere* (*absolutely*) *convergent and that* $\underline{\lim}_{n=\infty} (\lambda_n/n) \geq G > 0$. *Then if* L, $s = re^{i\bar{\theta}}$ $(r \geq 0)$, *be any ray issuing from the origin and lying in the left-hand half plane,* $\sigma \leq 0$, *either* L *is a half-line* J_1 *of* $f(s)$, *or else*

$$\lim_{m=\infty} |f(s_m)| = \infty$$

on every sequence of points $\{s_m = r_m e^{i\theta_m}\}$ *such that*

$$(20) \qquad \lim_{m=\infty} r_m = \infty, \quad \lim_{m=\infty} \theta_m = \bar{\theta}.$$

The proof rests on the inequality (10) of Section 2. 1. We show that if $|f(s)|$ is bounded on a sequence of points $\{s_m\}$ satisfying (20), then, no matter how small the positive number ζ, the family of functions

$$f_n(s) = f(2^n s) \qquad (n = 1, 2, \cdots)$$

is not normal in the region S,

$$1/2 < r < 1, \quad |\theta - \bar{\theta}| < \zeta \qquad (s = re^{i\theta}).$$

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* Cf. Pólya, *loc. cit.*, p. 559.

This, as in Theorem I, is sufficient to prove V, for under these circumstances there must be a point on L at which the family is not normal; thus L is a line J_1.

We observe that if the point $\bar{s} = \bar{\sigma} + i\bar{t}$ and the number η are so chosen that the circle C,

$$| s - \bar{s} | = \eta,$$

is entirely contained in S as well as in the left-hand half-plane, then

$$\overline{\lim_{n=\infty}} \; N_{f_n}(\bar{s}) = \infty, \tag{21}$$

where $N_{f_n}(\bar{s})$ is the maximum of $| f_n(s) |$ on C. In fact, we have

$$N_{f_n}(\bar{s}) = N_f(s_n), \tag{22}$$

where $N_f(s_n)$ is the maximum of $| f |$ on the circle C_n,

$$| s - s_n | = 2^n \eta, \quad s_n = 2^n \bar{s}.$$

Now the circles C_n are, for n great enough, of arbitrarily large radii; and thus, by the inequality (10) of Theorem I, we have, for all large values of n,

$$N_{f_n}(s_n) \geq A e^{-2^n \bar{\sigma}},$$

where A is independent of n. Thus, by (22), we get (21).

From (21) and the boundedness of $| f(s) |$ on the points $\{s_m\}$ it follows immediately that the family is not normal in S; and this completes the proof.

3. 2. The second of the two possibilities in the preceding theorem can be eliminated in certain cases. In particular, since

$$| f(it) | \leq \sum_{n=0}^{\infty} | a_n |,$$

and the series on the right converges, we have

THEOREM VI.* *Under the hypotheses of Theorem V, the upper and lower halves of the imaginary axis are half-lines J_1 of $f(s)$.*

3. 3. Another way of eliminating this possibility is to make certain hypotheses as to the order of $f(s)$. Thus:

THEOREM VII. *If the series* (1) *is everywhere* (*absolutely*) *convergent and* $\underline{\lim}_{r=\infty} (\lambda_{n+1} - \lambda_n) > 0$, *and if $f(s)$ is of non zero R-order, then every half line, issuing from the origin, and lying in the left-hand half-plane, is a line J_1 of $f(s)$.*

This theorem is a particular case of VI when the line coincides with the imaginary axis. Let us then consider a line L, $s = re^{i\bar{\theta}}$ $(r \geq 0)$, such that $\pi/2 < \bar{\theta} < 3\pi/2$, and the values of $| f(s) |$ in the angle S,

* It is readily verified that this conclusion holds for any exponential polynomial.

$$0 < r, \quad |\theta - \bar{\theta}| < \zeta,$$

where ζ is any arbitrarily small positive number. We shall show that

$$|f(s)| \leq 1$$

on a sequence of points $\{z_m\}$ lying in S and such that $|z_m|$ becomes infinite with m. Because ζ is arbitrary, this is, by Theorem V, sufficient to prove VII.

We prove firstly that

$$\varlimsup_{r=\infty} \{(\log r)^{-1} \log_2 M'(r, s)\} = \infty, \tag{23}$$

where $M'(r, s)$ is the maximum of $|f(s)|$ on the segment of the circle $|s| = r$ contained in S. For this we choose $0 < G \leq \varliminf_{r=\infty} (\lambda_{n+1} - \lambda_n)$ and employ the results and notation of the Lemma 2, Section 2. 4, with the exception that in this case we denote by s_n the point whose real part is σ_n and which lies on L. We then have for large values of r, rather than (19),

$$\begin{aligned} e^{\rho(1-\delta)\sigma_n} \log |h(s_n')| &\geq P e^{\rho(1-\delta)\sigma_n} \log |h(s_n)| \\ &\geq P \log \lambda_n \left[(\lambda_n \log \lambda_n)^{-1} \log (A_7 \lambda_n)^{-2} \right. \\ &\quad \left. + (\lambda_n \log \lambda_n)^{-1} |a_n| + \{\rho(1-\delta)\}^{-1} \right], \end{aligned}$$

where ρ is finite, positive and less than the R-order of $f(s)$. Hence

$$\lim_{n=\infty} [e^{\sigma_n' \rho(1-\delta)} \log |h(s_n')|] = \infty.$$

But

$$\sigma_n' \leq -|s_n'| b,$$

where b is the smaller of the two numbers $|\cos(\bar{\theta} - \zeta)|$, $|\cos(\bar{\theta} + \zeta)|$. Hence, since

$$|f(s_n')| \geq |h(s_n')| - B e^{\lambda_{n_0} |s_n'|},$$

where B is independent of n, we get

$$\varlimsup_{n=\infty} \{(\log |s_n'|)^{-1} \log_2 |f(s_n')|\} = \infty,$$

which justifies (23).

To complete the proof we now need only apply the theorem of Valiron for an angle, as stated in Section 2. 5 for we are thereby assured of the existence of the sequence of points $\{z_m\}$ lying in S on which $|f(s)|$ is bounded.

4. In conclusion we remark that we are not prepared to state whether it is possible for the second of the alternatives of Theorem I to be fulfilled. The same may be said for Theorem V. Another problem which suggests itself is that of determining under what conditions a Dirichlet series has a line $\bar{J}$.

ON SETS OF POLYNOMIALS AND ASSOCIATED LINEAR FUNCTIONAL OPERATORS AND EQUATIONS.*

By I. M. SHEFFER.†

I. INTRODUCTION.

Let $\{P_n(x)\}$, $(n = 0, 1, \cdot\cdot\cdot)$ be an infinitive sequence of polynomials such that $P_n(x)$ is of degree not exceeding n. As we shall be constantly dealing with sequences having this property, we shall refer to such a sequence, briefly, as a *set*. We consider in § II an algebra of sets of polynomials, each set being an element P. Associated with a set is a triangular infinite matrix, of the type used in summability theory for series; and the properties of sets are obtained from consideration of these matrices.

We consider integral and non-integral powers of a set, and apply the power relations to solving linear equations in sets. Then power series in sets are considered, and applied to implicit functions of sets and to the characteristic equation. Characteristic polynomials corresponding to a set are introduced, and in terms of them a canonical form is obtained for a set. Finally, we take up as a particular case the set M which corresponds to the Cesaro matrix (in divergent series theory), and determine conditions that a set P be commutative with M.

§ III is devoted to the analytic aspect of the theory of sets of polynomials. Each set P gives rise to a linear functional operator L (and hence to an equation), which can be expressed as a linear differential operator of infinite order. It possesses characteristic functions which are precisely the characteristic polynomials already introduced. Associated with L is a second linear operator, $\mathfrak{L}$, which possesses the same characteristic numbers as L, but whose characteristic functions are formal power series. *The treatment is formal.* In terms of the characteristic functions of L and $\mathfrak{L}$ we can obtain formal expansions of functions and solve formally the linear functional equations that arise.

In § IV we take up the systems of linear equations in infinitely many unknowns which arise from sets. By replacing characteristic polynomials by characteristic vectors, much of the preceding theory can be applied.

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† National Research Fellow.

It may be of some value to indicate the position of the present paper with respect to related publications. The algebraic aspect of sets is an attempt to generalize the permutability property enjoyed by the so-called Appell Polynomials.*

On the subject of linear differential equations of infinite order, it is necessary to make mention of the fundamental memoir by Bourlet,† where it is shown that linear operators are formally equivalent (in the field of analytic functions) to differential equations of infinite order, and where an equation for the "resolving" operator is obtained. Subsequent works on special equations have been published by von Koch, Perron, Hilb, H. T. Davis, the present writer, and others.‡

The use of the *adjoint equation* (that is made in § III) was suggested by the equation for the resolving kernel derived in a previous paper § for quite a different equation. This concept we believe to be useful in even more general linear differential equations than are considered here.

II. The Algebra of Sets.

1. *Integral Powers and Permutability.* Let $\{P_n(x)\}$ be a set of polynomials. We denote the set by P, and consider P as an element in an algebra subject to the two rules of operation:

(i) Addition: $P + Q$ is the set $\{P_n(x) + Q_n(x)\}$, $(n = 0, 1, \cdots)$.

(ii) Multiplication: Let

$$P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n, \quad Q_n(x) = q_{n0} + q_{n1}x + \cdots + q_{nn}x^n.$$

Then PQ is the set $\{PQ_n(x)\}$, $(n = 0, 1, \cdots)$, where

$$PQ_n(x) = p_{n0}Q_0(x) + p_{n1}Q_1(x) + \cdots + p_{nn}Q_n(x). \tag{1}$$

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* Appell, "Sur une classe de polynomes," *Annales Scientifiques de l'École Normale Supérieure*, (2), Vol. 9 (1880), pp. 119-144. In a later footnote we give the definition of these polynomials.

† C. Bourlet, "Sur les opérations en général et les équations différentielles linéaires d'ordre infini," *Annales de l'École Normale Supérieure*, (3), Vol. 14 (1897), pp. 133-190.

‡ References can be found in H. T. Davis, "Differential Equations of Infinite Order with Constant Coefficients," this Journal, Vol. 52 (1930), pp. 97-108; I. M. Sheffer, "Linear Differential Equations of Infinite Order, with Polynomial Coefficients of Degree One," *Annals of Mathematics*, Vol. 30 (1929), pp. 345-372.

§ I. M. Sheffer, "Expansions in Generalized Appell Polynomials, and a Class of Related Linear Functional Equations," *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 261-280.

It is easily seen that the usual laws of association and distribution hold, as does the commutative law for addition; and that in general multiplication is *not* commutative.

Definition. If $PQ = QP$ we shall say that the sets P and Q are *permutable.** A necessary and sufficient condition for permutability is given, in terms of the coefficients p_{ij}, q_{ij}, by expanding PQ and QP and equating like powers of x.

Given a set $P: P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n$, $(n = 0, 1, \cdots)$. With P we associate the triangular infinite matrix $M_P: \| m_{nk} \|$, where $m_{nk} = p_{nk}$, $(k = 0, 1, \cdots, n;\ m_{nk} = 0,\ k > n)$; and we observe that

$$P + Q \text{ in sets becomes } M_{P+Q} = M_P + M_Q \text{ in matrices;}$$
$$PQ \text{ in sets becomes } M_{PQ} = M_P \cdot M_Q \text{ in matrices.}$$

The algebra of sets is, then, essentially the algebra of the associated infinite matrices; and we can rewrite the properties of matrices in terms of sets.

Notation. $I: I_n(x) = x^n$ is the *identity* set; $D: D_n(x) = d_n x^n$ is a *diagonal* set;

$S: S_n(x) = sx^n$ is a *scalar* set.

I, S, and 0 are the only sets permutable with every set.

Definition. For every positive † integer i, $P^i = PP^{i-1}$ is the i-th power of P, and we write $P^i: \{P_n{}^i(x)\}$, $(n = 0, 1, \cdots)$.

P^i is uniquely determined from P, and is permutable with P.

Definition. $P^{-1}: \{P_n{}^{-1}(x)\}$ is defined by $I = P^0 = P(P^{-1})$; and $P^{-k}: \{P_n{}^{-k}(x)\}$ by $P^{-(k-1)} = P(P^{-k})$.

Definition. P is *non-singular* if $P_n(x)$ is of degree exactly n, $(n = 0, 1, \cdots)$; otherwise *singular.*

$P^{-1}, P^{-2}, \cdots$ exist if and only if P is non-singular; and when this is so, $P^{-1}, P^{-2}, \cdots$ are uniquely determined, and $P_n{}^{-k}(x)$ is of degree exactly n. Further, P^i, P^j are permutable for all integers i, j. Some immediate corollaries are: ‡

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* An interesting system of permutable sets is furnished by the Appell Polynomials: If $P_n(x)$, $Q_n(x)$ are defined by

$$A(t)e^{tx} = \sum_0^\infty P_n(x)t^n/n!, \qquad B(t)e^{tx} = \sum_0^\infty Q_n(x)t^n/n!,$$

then P and Q are permutable, and, in fact, the set PQ is given by

$$A(t)B(t)e^{tx} = \sum_0^\infty PQ_n(x)t^n/n!$$

† $P^0 \equiv I$.

‡ When negative powers are used it is assumed that P is non-singular.

(i) $\sum_{i=-l}^{m} \lambda_i P^i$ and $\sum_{j=-s}^{t} \mu_j P^j$ are permutable.*

(ii) $P^i P^j = P^{i+j}$, i, j any integers; and $(P^i)^j = (P^j)^i = P^{ij}$.

(iii) If P is non-singular then the only set Q satisfying the equation $PQ = 0$ is $Q = 0$.

(iv) If P is non-singular then a necessary and sufficient condition that Q and R be permutable is that $PQR = PRQ$.

(v) If P and Q are non-singular then $(PQ)^{-1} = Q^{-1}P^{-1}$.

(vi) If P and Q are permutable, so are P^i and Q^j, i, j any integers; and so are † $\sum_{i=-l}^{m} \lambda_i P^i$, $\sum_{j=-s}^{t} \mu_j Q^j$; and $\sum_{i,j=-l,-s}^{m,t} \lambda_{ij} P^i Q^j$ is permutable to P (and to Q).

(vii) If p_{nn} is the coefficient of x^n in $P_n(x)$, then $p_{nn}{}^i$ is the coefficient of x^n in $P_n{}^i(x)$, i any integer.

2. *Non-Integral Powers.* Till now we have considered only positive and negative integral powers. It is however desirable to have the complete continuum (real or complex) of powers.

Definition. The numbers $\{p_n = p_{nn}\}$, $(n = 0, 1, \cdots)$ in $P_n(x) = p_{n0} + \cdots + p_{nn}x^n$ are the *characteristic numbers* ‡ for P.

COROLLARY. *The characteristic numbers for P^i are $\{p_n{}^i\}$, i any integer; and if $s(\lambda)$ is any polynomial, the set $s(P)$ has the characteristic numbers $s(p_n)$. Moreover, if P, Q have the characteristic numbers $\{p_n\}$, $\{q_n\}$, then PQ and QP have the characteristic numbers $\{p_n q_n\}$.*

Definition. P is *complete* § if $p_m \neq p_n$, $(m \neq n)$.

Definition. Let σ be any number (real or complex). Then P^σ is any set which satisfies the two following conditions:

(i) P^σ is permutable with P;

(ii) the characteristic numbers $\{p_n{}^{(\sigma)}\}$ for P^σ are given by ¶ $p_n{}^{(\sigma)} = p_n{}^\sigma$.

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* These series may extend to infinity in both directions, provided convergence takes place. We shall consider this point presently.

† See preceding footnote.

‡ The reason for this name will appear. It should not be overlooked that the characteristic numbers form an ordered set of numbers.

§ Our use of the property of "completeness" is such that its definition can be widened to include a more general class of sets. We shall return to this remark in a later footnote.

¶ We may agree to choose the same determination (for all n) of the logarithm in $P_n{}^\sigma = e^{\sigma \log p_n}$.

We observe that for $\sigma = i =$ integer, this definition coincides with that one already given for P^i.

LEMMA. *Let P be complete, and let $\{q_n\}$ be an arbitrary set of numbers. There exists a unique set Q with characteristic numbers $\{q_n\}$ which is permutable with P.*

The lemma follows from a consideration of the equations resulting when one equates PQ with QP.

THEOREM. *If P is complete there exists at least one set P^σ for every σ whose real part is positive; and if in addition, P is non-singular there is at least one P^σ for all σ.*

Both parts of the theorem follow from the preceding lemma. That there may exist more than one P^σ is a consequence of the many-valuedness of $p_n{}^\sigma$ for $\sigma \neq$ integer.

LEMMA. *The properties $P^\alpha P^\beta = P^{\alpha+\beta}$, $(P^\alpha)^\beta = P^{\alpha\beta}$* hold for all * α, β.

3. *Application to Equations in Sets.*† Let $s(\lambda)$ be the polynomial

$$s(\lambda) = s_1\lambda + s_2\lambda^2 + \cdots + s_k\lambda^k;$$

and let P be a givn set. We seek a set X which satisfies the equation ‡

$$s_1X + s_2X^2 + \cdots + s_kX^k = P. \tag{a}$$

Let the characteristic numbers of P be $\{p_n\}$. If X exists then P, being a polynomial in X, is permutable with X. Also, $\{x_n\}$ being the characteristic numbers of X,

$$s_1x_n + s_2x_n{}^2 + \cdots + s_kx_n{}^k = p_n, \qquad (n = 0, 1, \cdots). \tag{b}$$

Conversely, any set X, which is permutable with P and whose characteristic numbers x_n satisfy (b), is a solution of (a). By a preceding lemma, if P is complete such an X can be found. Hence

THEOREM. *X is a solution of* (a) *if and only if it is permutable with P and has its characteristic numbers $\{x_n\}$ satisfy* (b). *If P is complete, then there exists at least one solution X.*

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* It is to be understood that when any determination of the many-valued functions involved has been made on one side of the equation, then a suitably chosen determination on the other side will make the equation a true one.

† The methods here used apply equally well to equations in square matrices (say of order n), when the given matrix has n linearly independent invariant directions.

‡ If $s(\lambda)$ contains a constant term s_0, we can reduce the case to (a) by transposing s_0I to the right.

In a similar manner we can treat the set-equation

$$L_1X + L_2X^2 + \cdots + L_kX^k = P, \tag{c}$$

where $L_1, \cdots, L_k$ are permutable with P. Letting $l_n^{(i)}$, p_n, x_n be the characteristic numbers of L_i, P, X, we obtain the

THEOREM. *If X is a solution of* (c) *then the x_n satisfy*

$$l_n^{(1)}x_n + l_n^{(2)}x_n^2 + \cdots + l_n^{(k)}x_n^k = P_n, \qquad (n = 0, 1, \cdots). \tag{d}$$

Moreover, if P is complete, there exists at least one solution X which is permutable with P.

And we can equally well consider the set-equation

$$\sum_{i_1, \ldots, i_k=0}^{r} L_{i_1 \ldots i_k} X_1^{i_1} X_2^{i_2} \cdots X_k^{i_k} = P \tag{e}$$

in the k unknown sets $X_1, \cdots, X_k$, where there is no term independent of the X_i, and where the $L_{i_1 \ldots i_k}$ are sets all permutable with P.

THEOREM. *If a solution $X_1, \cdots, X_k$ exists, then the characteristic numbers $x_n^{(l)}$, $(l = 1, \cdots, k)$ satisfy the equations*

$$\sum_{i_1, \ldots, i_k=0}^{r} l_n^{(i_1 \ldots i_k)} x_n^{(1)i_1} x_n^{(2)i_2} \cdots x_n^{(k)i_k} = p_n, \qquad (n = 0, 1, \cdots); \tag{f}$$

and if P is complete, there exists at least one solution $X_1, \cdots, X_k$, each X_i permutable with P.

4. *Power Series in Sets; Characteristic Equation.* Let $Q^{(0)}, Q^{(1)}, \cdots$ be an infinite sequence of sets. We may then form the series

$$Q = \sum_{k=0}^{\infty} Q^{(k)}, \tag{1}$$

and ask when this symbol is itself a set.

Definition. The series (1) converges if

$$q_{ni} = \lim_{s=\infty} \sum_{k=0}^{s} q_{ni}^{(k)} \quad \text{exists,} \quad \begin{cases} n = 0, 1, 2, \cdots \\ i = 0, 1, \cdots, n \end{cases};$$

and the *sum* is the set

$$Q: \{Q_n(x) = q_{n0} + q_{n1}x + \cdots + q_{nn}x^n\}.$$

If $Q^{(k)} = f_kP^k$, where the f_k's are scalars, we have a *power series* in the set P:

$$f(P) = \sum_{k=0}^{\infty} f_k P^k.$$

Let M be a square matrix of order n, and let its n characteristic numbers be $\lambda_1, \cdots, \lambda_n$. Then it is known that

LEMMA. *If $\lambda_1, \cdots, \lambda_n$ are distinct, a necessary and sufficient condition that $\sum_{k=0}^{\infty} f_k M^k$ converge is that the series $f(z) = \sum_0^{\infty} f_k z^z$ converge for $z = \lambda_1, \cdots, \lambda_n$; and if $\lambda_1, \cdots, \lambda_n$ are not necessarily distinct a sufficient condition for convergence is that for each distinct λ (of order s, say), the series for $f(\lambda), f'(\lambda), \cdots, f^{(s-1)}(\lambda)$ shall converge.*

To apply this lemma we make the following observations:

If P is any set, its associated matrix M_P has the characteristic numbers $p_0, p_1, \cdots$. If we denote by M_{P_n} the matrix consisting of the first $n+1$ rows and columns of M_P, then M_{P_n} has the characteristic numbers $p_0, \cdots, p_n$, and to it the lemma applies. Furthermore, the operations $P + Q$, PQ or their equivalents $M_{P+Q} = M_P + M_Q$, $M_{PQ} = M_P \cdot M_Q$ give rise to the operations $M_{(P+Q)_n} = M_{P_n} + M_{Q_n}$, $M_{(PQ)_n} = M_{P_n} \cdot M_{Q_n}$. As we let n tend to infinity, then, there results the

THEOREM. *If P is complete, a necessary and sufficient condition that the series $f(P) = \sum_0^{\infty} f_k P^k$* converge *is that* $\sum_0^{\infty} f_k z^k$ converge for $z = p_n$, *$(n = 0, 1, \cdots)$; and for any set P a sufficient condition for convergence is that for each distinct p_i (say of order* s_i), the series $f(p_i)$, $f'(p_i)$, $\cdots, f^{(s_i-1)}(p_i)$ shall converge.*

COROLLARY. *If $f(P) = \sum_0^{\infty} f_k P^k$ converges, its characteristic numbers are $\{f(P_n)\}$.*

COROLLARY. *$f(P)$ is permutable with P.*

Let $f(z) = \sum_0^{\infty} r_n(z) = \sum_0^{\infty} s_n(z)$ be two expansions for the analytic function $f(z)$. If $\sum_0^{\infty} r_n(A)$ and $\sum_0^{\infty} s_n(A)$ both converge one may ask if they have the same sum-set. But first: what are we to mean by $r_n(A)$, $s_n(A)$? If $r_n(z)$ and $s_n(z)$ are polynomials there is no difficulty. But if they are more general analytic functions, $r_n(A)$ and $s_n(A)$ must be defined. We make the Definition: Let $f(z)$ be an analytic function. Then $f(P)$ is any set satisfying the conditions

* $s_i = \infty$ is permissible.

(i) $f(P)$ is permutable with P;
(ii) the characteristic numbers f_n of $f(P)$ are related to the p_n of P by $f(p_n) = f_n$, $(n = 0, 1, \cdots)$.

It is to be noted that given $f(z)$, $f(P)$ need not exist for every P, and for two reasons: (1) some p_n may be outside the region of existence of $f(z)$; (2) even if the $f(p_n)$ exist, it does not follow that a set with these characteristic numbers will exist and be permutable with P. Further, if $f(z)$ is multiple-valued, $f(P)$ if it exists, will not be unique.

With this definition we can now establish the

THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} r_k(z)$ *converge at* $z = p_n$ $(n = 0, 1, \cdots)$, *where* $f(z)$ *and the* $r_k(z)$ *are analytic functions. Further, let* $\sum_{k=0}^{\infty} r_k(P)$ *converge. Then its sum-set is* $f(P)$.

Proof. Let Q be the sum-set. $r_k(P)$ is, by definition, permutable with P. We then find by direct multiplication that $QP = PQ$. Again, the characteristic numbers of $r_k(P)$ are $\{r_k(p_n)\}$, so that $q_n = \sum_{k=0}^{\infty} r_k(p_n)$. But this is $f(p_n)$. Hence $Q = f(P)$.

Two interesting examples of power series are e^P and $\log(1 + P)$:

$$e^P = I + P/1! + P^2/2! + \cdots; \quad \log(I + P) = P - P^2/2 + P^3/3 - \cdots.$$

If P and Q are permutable then we have the functional equation $e^P \cdot e^Q = e^{P+Q}$.

Let $f(x, y)$ be analytic in (x, y) in a region about $(0, 0)$, and suppose $f(0, 0) = 0$, $\partial f(0, 0)/\partial y \neq 0$. By the implicit function theorem there exists a function $y(x) = \sum_{n=0}^{\infty} y_n x^n$ analytic about $x = 0$, with $y(0) = 0$, which makes $f(x, y) \equiv 0$ in x. Now let P be a variable set, and consider the set-equation

$$\text{(a)} \qquad f(P, Q) = 0,$$

in the unknown set Q. There exists a solution in the form $Q = \sum_{n=0}^{\infty} y_n P^n$ provided the characteristic numbers of P are sufficiently small; and Q is permutable with P.

Given a set P. To its components $P_n(x)$ there correspond finite characteristic equations; and to P itself, under suitable convergence conditions, there corresponds a transcendental characteristic equation. The precise relations are given by the two following theorems.

THEOREM. *Let*

$$(1) \qquad \Delta_n(\lambda) = (\lambda - p_0)(\lambda - p_1) \cdots (\lambda - p_n), \qquad (n = 0, 1, \cdots);$$

then

(2) $$\Delta_n(P_n)=0.$$

that is, if

$$\Delta_n(\lambda)=\delta_{n0}+\delta_{n1}\lambda+\cdots+\delta_{n,n+1}\lambda^{n+1},$$

then *

(2′) $$\delta_{n0}P_n{}^0(x)+\delta_{n1}P_n{}^1(x)+\cdots+\delta_{n,n+1}P_n{}^{n+1}(x)=0, \qquad (n=0,1,\cdots).$$

For: Let M_{P_n} denote again the matrix of the first $n+1$ rows and columns of M_P. Then (2′) is a restatement of the fact that M_{P_n} satisfies its matrix equation.

COROLLARY. *We also have*

$$\delta_{n0}P_i{}^0(x)+\delta_{n1}P_i{}^1(x)+\cdots+\delta_{n,n+1}P_i{}^{n+1}(x)=0, \qquad (i=0,1,\cdots,n).$$

THEOREM. *Let the complete set P have the characteristic numbers $\{p_n\}$. If an analytic function $\Delta(z)=\sum_0^\infty \delta_k z^k$ exists such that the series for $\Delta(p_n)$, $(n=0,1,\cdots)$ all converge, and such that $\Delta(z)$ vanishes at and only at p_n, then P satisfies the "characteristic" equation*

$$\Delta(P)=0: \quad \sum_0^\infty \delta_k P^k=0.$$

Proof. By a previous theorem $\Delta(P)=\sum_0^\infty \delta_k P^k$ converges, and it is permutable with P. Further, its characteristic numbers are $\Delta(p_n)=0$, whence from the fact that P is complete follows that $\Delta(P)=0$.

It may happen that $\Delta(z)$ vanishes at $z=p_n$ but that the power series for $\Delta(z)$ diverges at $z=p_n$. We cannot then use the power series. But if, for example, the set of numbers $\{p_n\}$ lies in a simply-connected region in which $\Delta(z)$ is analytic, we can expand $\Delta(z)$ in a series of polynomials, convergent in this region:

$$\Delta(z)=\sum_0^\infty G_n(z);$$

and then P satisfies the characteristic equation $\sum_0^\infty G_n(P)=0$.

5. *Characteristic Polynomials; Canonical Form.* Associated with a set P is a set Θ which is of importance in questions of permutability, and in terms of which a canonical form for P is possible. It has, moreover, an invariantive character under an operation soon to be considered, and we term

* It is to be recalled that $P_n{}^i(x)$ is the $(n+1)$st polynomial in the set P^i.

it the *characteristic set* for P; its polynomial components $\Theta_n(x)$ are the *characteristic polynomials.*

Let $\{p_n\}$ be the characteristic numbers for the set P, and define P^* as the diagonal set $\{P_n^*(x) = p_n x^n\}$. Then

Definition. Θ is a *characteristic set* for P if it satisfies the two conditions:

(i) $\Theta P = P^*\Theta$;
(ii) $\Theta_n(x)$ is of degree n, so that Θ is non-singular.

It is to be observed that if Θ is a characteristic set, so is $C\Theta$ where C is any diagonal set all of whose characteristic numbers are different from zero. But this is the maximum of arbitrariness, and is for our purpose unessential:

THEOREM.† *If P is complete there exists a characteristic set Θ which is unique to within an arbitrary diagonal-set multiplier on the left with non-zero characteristic numbers.*

Proof. If we equate the n-th polynomials on both sides of (i) we obtain a set of equations for the coefficients of $\Theta_n(x)$. These equations permit θ_{nn} to be arbitrary, after which $\theta_{n,n-1}, \cdots, \theta_{n0}$ are uniquely determined. The arbitrariness of θ_{nn} ($\neq 0$) allows for multiplication of Θ by a diagonal set on the left.

By its definition, Θ is non-singular, and Θ^{-1} exists. Hence

THEOREM. *If P is complete it possesses the canonical form*

$$P = \Theta^{-1}P^*\Theta. \tag{1}$$

We come now to a theorem on permutability:

THEOREM. *Let P be complete and Θ its characteristic set. A necessary and sufficient condition that Q be permutable with P is that Q possesses Θ as a characteristic set; and Q has then the canonical form $Q = \Theta^{-1}Q^*\Theta$.*

Necessity: Given $PQ = QP$. Then $\Theta PQ = \Theta QP$; that is, $P^*\Theta Q = \Theta QP$, or

$$(\Theta Q)P = P^*(\Theta Q). \tag{a}$$

In other words, the set ΘQ satisfies the first condition for a characteristic set.

† It was remarked in a previous footnote that the definition of completeness can be enlarged. Thus, to have the set Θ exist it is not necessary that $p_m \neq p_n$, $m \neq n$. What is necessary is that for each n, the matrix M_{P_n} (of order $n+1$) have $n+1$ linearly independent invariant directions. The definition of completeness could be given to be equivalent to this property, and most of the subsequent results would continue to hold.

If we rewrite (a) in terms of the polynomial components, we find that $(\Theta Q)_n(x)$ must be a multiple of $\Theta_n(x)$, so that (b) $\Theta Q = D\Theta$, D being a diagonal set. Now $\theta_{nn} \neq 0$ so that on equating the coefficients of x^n in the n-th polynomial of (b) we have $d_n = q_n$, or $D = Q^*$.
Sufficiency: Given $Q = \Theta^{-1}Q^*\Theta$. Then

$$\Theta PQ = P^*\Theta Q = P^*Q^*\Theta, \qquad \Theta QP = Q^*\Theta P = Q^*P^*\Theta.$$

But † $P^*Q^* = Q^*P^*$, whence by operating on both equations with Θ^{-1} on the left we obtain $PQ = QP$.

THEOREM. *If P possesses a characteristic set Θ, and $s(\lambda)$ is a polynomial, then $s(P)$ has the canonical form $s(P) = \Theta^{-1}s(P^*)\Theta$. Further, if $f(P) = \sum_{k=0}^{\infty} f_k P^k$ converges, then $f(P) = \Theta^{-1}f(P^*)\Theta$.*

6. *The Cesaro Set, and Permutability.*‡ Let M_P be the triangular infinite matrix associated with P:

$$M_P: \left\| \begin{matrix} p_{00} & & & \\ p_{10} & p_{11} & & \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ p_{n0} & p_{n1} & \cdots & p_{nn} \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{matrix} \right\| .$$

Such matrices are of some interest in the summability theory of divergent series; they give rise to the Silverman-Toeplitz methods of summation. Perhaps the most important method both historically and practically is the Cesaro method of arithmetic means. Its matrix is $\| m_{nk} \|$, $m_{nk} = 1/n+1$, $(k = 0, 1, \cdots, n;\ m_{nk} = 0,\ k > n)$, and the corresponding set of polynomials $M: \{M_n(x)\}$ is given by

$$M_n(x) = (1 + x + \cdots + x^n)/(n+1).$$

We shall obtain the canonical form for M, and shall consider the condition for permutability § with M.

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† We use the LEMMA. *All diagonal sets are permissible.*

‡ In this section we usually omit proofs; they are easily provided.

§ In divergent series theory, permutability is of interest, for if two "regular" methods of summation are permutable, they give the same sum to any sequence which both sum.

From the standpoint of matrices, a characterization of sets (that is, matrices) permutable with M has been given by Hausdorff, *Mathematische Zeitschrift*, Vol. 9 (1921), pp. 74-109; pp. 280-299.

Let P be a set, and $\{p_n\}$ its characteristic numbers. On equating PM with MP we obtain the

THEOREM. *A necessary and sufficient condition that P be permutable with M is that*

(A) $P_n(x) = p_n x^n - \binom{n}{1}\Delta p_n x^{n-1} + \binom{n}{2}\Delta^2 p_n x^{n-2} - \cdots + (-1)^n \binom{n}{n}\Delta^n p_n$, *where* $\Delta p_n = p_n - p_{n-1}$, $\Delta^2 p_n = \Delta(\Delta p_n)$, *etc.*

Further results on permutability are:

THEOREM. *P is permutable with M if and only if*

(B) $$P_n(x) = p_n(x-1)^n + \binom{n}{1} p_{n-1}(x-1)^{n-1} + \cdots + \binom{n}{n} p_0.$$

COROLLARY. *P^s is permutable with M, and*

$$P_n{}^s(x) = p_n{}^s(x-1)^n + \binom{n}{1} p^s{}_{n-1}(x-1)^{n-1} + \cdots + \binom{n}{n} p_0{}^s.$$

THEOREM. *Let $P(t)$ be the formal power series $P(t) \sim \sum_0^\infty p_n t^n/n!$ A necessary and sufficient condition that P be permutable with M is that*

(C) $$e^t P[t(x-1)] \sim \sum_0^\infty P_n(x) t^n/n!$$

Definition. $P(t) \sim \sum_0^\infty p_n t^n/n!$ is the *generating function* for the set $P: \{P_n(x)\}$.

COROLLARY. *If ${}_kP(t)$ is the generating function for M^{-k}, then*

$$_kP(t) = (d/dt)\{t \cdot {}_{(k-1)}P(t)\},$$

and

(1) $$_kP(t) = e^t S_k(t)$$

where $S_k(t)$ is a polynomial of degree k defined by

(2) $$S_0(t) = 1, \qquad S_k(t) = (t+1)S_{k-1}(t) + tS'_{k-1}(t).$$

Furthermore,

(3) $$t^n = b_{n0}S_0(t) + b_{n1}S_1(t) + \cdots + b_{nn}S_n(t)$$

where the b_{ni} are given by

(4) $R_n(t) \equiv 1(t-1)(t-2)\cdots(t-n) = b_{n0} + b_{n1}t + \cdots + b_{nn}t^n$;

and

(5) $$S_n(t) = a_{n0} + a_{n1}t + \cdots + a_{nn}t^n$$

where the a_{ni} satisfy

$$t^n = a_{n0}R_0(t) + a_{n1}R_1(t) + \cdots + a_{nn}R_n(t). \tag{6}$$

Letting R, S denote the sets $\{R_n(x)\}$, $\{S_n(x)\}$, we see that

$$(RS)_n(t) = (SR)_n(t) = t^n = I_n(t).$$

That is,

COROLLARY. *R and S are inverse sets: $R = S^{-1}$, $S = R^{-1}$.*

From the recurrence relation (2) for $S_n(t)$ we can establish the

LEMMA. *The zeros of $S_n(t)$ are real, negative, and simple, and they separate the zeros of $S_{n-1}(t)$.*

THEOREM. *If P and Q are each permutable with M, then P and Q are permutable.*

Conditions (A), (B) and (C) for permutability with M involve the characteristic numbers $\{p_n\}$ of P. From (C) we obtain a further necessary and sufficient condition:

$$P_0(x) - \binom{n}{1}P_1(x) + \cdots + (-1)^n\binom{n}{n}P_n(x) = (-1)^n p_n (x-1)^n; \tag{D}$$

and from this we get a condition *independent* of the characteristic numbers:

THEOREM. *All sets P permutable with M (and only those) satisfy the relations* *

$$\begin{aligned}(x-1)\{P_0'(x) - \tbinom{n}{1}P_1'(x) + \cdots + (-1)^n\tbinom{n}{n}P_n'(x)\}& \\ - n\{P_0(x) - \tbinom{n}{1}P_1(x) + \cdots + (-1)^n\tbinom{n}{n}P_n(x)\} &= 0, \\ (n = 0, 1, \cdots).&\end{aligned} \tag{E}$$

In divergent series theory, M and its positive integral powers (the Cesaro means of positive integral order) are very important, whereas the negative powers are useless since they give rise to summation methods which do not sum all convergent series. It is therefore rather remarkable that every set P which is permutable with M (and this includes M, M^2, $M^3, \cdots$) can be expressed as a series of polynomials in M^{-1}:

THEOREM. *Let P be a set permutable with M, and $\{p_n\}$ its characteristic numbers. Let $\{q_n\}$ be defined by*

$$q_n = \frac{p_0}{0!}\,\frac{(-1)^n}{n!} + \frac{p_1}{1!}\,\frac{(-1)^{n-1}}{(n-1)!} + \cdots + \frac{p_n}{n!}\,\frac{1}{0!}.$$

Then P is given by the following convergent series of polynomials in M^{-1}:

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* In (E), primes denote differentiation.

$$P = \sum_{n=0}^{\infty} q_n I(M^{-1}-1)(M^{-1}-2)\cdots(M^{-1}-n).$$

Proof. Set $T^{(n)} = I(M^{-1}-1)\cdots(M^{-1}-n)$. On expanding we have

$$T^{(n)} = b_{n0}I + b_{n1}M^{-1} + \cdots + b_{nn}M^{-n},$$

so that the generating function for $T^{(n)}$ is

$$T^{(n)}(t) = b_{n0}\cdot P_0(t) + \cdots + b_{nn}\cdot {}_nP(t) = t^n e^t.$$

Now let $L_i(t) \sim \sum_{n=0}^{\infty} l_{in}t^n$, $(i = 0, 1, \cdots)$, and suppose $\sum_{i=0}^{\infty} l_{in} = l_n$ exists, $(n = 0, 1, \cdots)$; and set $L(t) \sim \sum_{0}^{\infty} l_n t^n$. Further, let

$$e^t L_i[t(x-1)] \sim \sum_{n=0}^{\infty} L_{i,n}(x)t^n/n!, \qquad e^t L[t(x-1)] \sim \sum_{n=0}^{\infty} L_n(x)t^n/n!.$$

Then

$$L_n(x) = \sum_{i=0}^{\infty} L_{in}(x), \qquad (n = 0, 1, \cdots),$$

and each series is a *convergent* series.

Applying this, we see that $K = \sum_{n=0}^{\infty} q_n T^{(n)}$ will converge if in the formal sum $\sum_{n=0}^{\infty} q_n T^{(n)}(t)$, the coefficients of $1, t, t^2, \cdots$ all exist. But this is true since

$$\sum_{0}^{\infty} q_n T^{(n)}(t) \sim \sum_{n=0}^{\infty} t^n [q_0/n! + q_1/(n-1)! + \cdots + q_n/0!] \sim \sum_{n=0}^{\infty} p_n t^n/n!.$$

Hence the sum-set K exists and its generating function is $\sum_{0}^{\infty} p_n t^n/n!$. Since this is also the generating function for P, we must have $K = P$.

The characteristic set for M is a very simple set:

THEOREM. *M has the characteristic set*

$$\Theta:\quad \Theta_n(x) = (x-1)^n.$$

COROLLARY. *The set P, with characteristic numbers $\{p_n\}$ is permutable with M if and only if*

$$P = \Theta^{-1}P^*\Theta,$$

where $P^*: \{P_n^*(x) = p_n x^n\}$ and Θ is given above.

We can now give a new proof of the second preceding theorem: M^{-1} has the characteristic numbers $\{n+1\}$, $(n = 0, 1, \cdots)$, whence $T^{(n)}$ has the

characteristic numbers $\{1 \cdot k(k-1) \cdots (k-n+1)\}$, $(k=0, 1, \cdots)$. Therefore $K = \sum_0^\infty q_n T^{(n)}$ has the characteristic numbers

$$\{\sum_{n=0}^\infty q_n \cdot 1(k-1) \cdots (k-n)\},$$
$$= \{k!\,[q_0/k! + q_1/(k-1)! + \cdots + q_k/0!\,]\} = \{p_k\}, \qquad (k=0,1,\cdots).$$

That is, the K-series converges, and $K = P$.

III. Linear Operators and Equations Associated with Sets.

1. *Introduction.* We have considered in § II the algebraic aspects of sets. Going farther, it is possible to treat sets of polynomials analytically. In so doing we are led to associate with each set a definite linear operator; and this pair: the set of polynomials and the operator in turn give rise to a second operator and set of functions (no longer polynomials). We examine the relations existing between these pairs, and in general consider the formal solution of the functional equations on the two operators. *The treatment is formal.**

2. *The Linear Operator L.* Let P be a given set:

$$P\colon\ P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n, \qquad (n=0,1,\cdots).$$

We wish to determine a linear operator L which will transform polynomials into polynomials and in such manner that the identity set I is carried into P:

$$L[x^n] = P_n(x), \qquad (n=0,1,\cdots). \tag{1}$$

Such an operator may be put in various forms; for our purpose it will be convenient to express L as a differential expression of infinite order.

Theorem. *The linear operator*

$$L \equiv L[y(x)] = \sum_{n=0}^\infty L_n(x) y^{(n)}(x), \tag{2}$$

where $L_n(x)$ is the polynomial

$$L_n(x) = (1/n!)\,[P_n(x) - \tbinom{n}{1} x P_{n-1}(x) + \tbinom{n}{2} x^2 P_{n-2}(x) - \cdots + (-1)^n x^n P_0(x)], \tag{3}$$

carries I into P.

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* Conditions under which the formal processes can be shown to hold rigorously are reserved for another occasion.

Proof. The method of induction can be used. Or better: *define* the formal power series

$$(4) \qquad P(t;x) \sim \sum_{n=0}^{\infty} P_n(x) t^n/n!,$$

$$(5) \qquad L(t;x) \sim \sum_{n=0}^{\infty} L_n(x) t^n.$$

The right hand member of (3) is the coefficient of t^n in the expansion of $e^{-tx}P(t;x)$, so that on multiplying (3) by t^n and summing we obtain

$$(6) \qquad L(t;x) \sim e^{-tx}P(t;x).$$

Again, from (2),

$$(7) \qquad L[e^{tx}] \sim e^{tx}L(t;x) \sim P(t;x);$$

and also

$$L[e^{tx}] \sim \sum_{0}^{\infty} (t^n/n!) L[x^n].$$

On comparing this with (7) we have $L[x^n] = P_n(x)$.

LEMMA. *If L is the operator (2), and Q is any set, then*

$$L[Q] = QP; \text{ i. e., } L[Q_n(x)] = (QP)_n(x), \qquad (n = 0, 1, \cdots).$$

COROLLARY. *If L_P, L_Q are the operators for P, Q, then L_P, L_Q are commutative if and only if P and Q are permutable.*

In the study of linear operators one usually seeks the characteristic functions; that is, those functions which repeat themselves (with a constant multiplier) under the operation. For L we have the

THEOREM. *The linear differential equation of infinite order*

$$(8) \qquad L[u(x)] - \lambda u(x) = 0$$

has a polynomial solution if and only if λ is one of the characteristic numbers $\{p_n\}$ of P; and if $p_n \neq p_i$ $(i \neq n)$, then to $\lambda = p_n$ corresponds a unique polynomial, which is of degree exactly n.*

Proof. On substituting $u(x) = u_0 + u_1 x + \cdots + u_s x^s$ into (8) one finds for $u_0, \cdots, u_s$ a set of $s + 1$ linear homogeneous equations, the determinant of which vanishes only when $\lambda = p_0, p_1, \cdots, p_s$. When λ has one of these values there is at least one solution. If for all i different from n, $p_i \neq p_n$, then to $\lambda = p_n$ corresponds just one solution, and it is of degree $s = n$.

If some of the characteristic numbers $\{p_n\}$ are equal there need not

* There is however an arbitrary constant multiplier.

exist a polynomial solution of (8) for every degree $n = 0, 1, \cdots$. This is undesirable, so we limit ourselves to the case $p_m \neq p_n$, $m \neq n$.

COROLLARY. *If P is complete there exists one and only one polynomial solution of* (8) *for every degree* $n = 0, 1, \cdots$.

Moreover,

THEOREM. *If P is complete the polynomial solutions of* (8) *form the characteristic set for P.*

Proof. Let Θ denote the characteristic set for P. Then $\Theta P = P^*\Theta$. But $L[\Theta] = \Theta P$ and $(P^*\Theta)_n(x) = p_n\Theta_n(x)$, so that $L[\Theta_n(x)] = p_n\Theta_n(x)$.

Let P be non-singular. Then P^{-1} exists, and if L^{-1} is the linear operator for P^{-1}:

(a) $$L^{-1}[x^n] = P_n^{-1}(x),$$

we have

LEMMA. *$L^{-1}L = LL^{-1}$ is the identity operator; that is,*

(b) $$L^{-1}[P_n(x)] = x^n, \qquad L[P_n^{-1}(x)] = x^n.$$

COROLLARY. *A formal solution of the functional equation*

$$L[y(x)] = f(x)$$

is

$$y(x) = L^{-1}[f(x)].$$

In general, however, this expression does not converge.

If $P = D$ is a diagonal set: $D_n(x) = d_nx^n$, L assumes a simple form:

$$L[y(x)] \equiv \sum_{n=0}^{\infty} \sigma_n x^n y^{(n)}(x),$$

where

$$\sigma_n = (1/n!)\left[d_n - \binom{n}{1} d_{n-1} + \cdots + (-1)^n d_0\right].$$

THEOREM. *If P is complete, and L_P, L_θ, L_{P^*}, $L_{\theta^{-1}}$ are the operators for P, Θ, P^*, Θ^{-1}, then*

$$L_P = L_\theta L_{P^*} L_{\theta^{-1}},$$

the operations being performed from left to right.

3. *The Linear Operator $\mathfrak{L}$.* We have defined $P(t; x)$ and $L(t; x)$ in (4) and (5) as formal power series in t. If we rewrite them as power series in x, we have

$$(4') \qquad P(t;x) \sim \sum_{n=0}^{\infty} \mathcal{P}_n(t)x^n/n!,$$

$$(5') \qquad L(t;x) \sim \sum_{n=0}^{\infty} \mathcal{L}_n(t)x^n,$$

where $\mathcal{P}_n(t)$, $\mathcal{L}_n(t)$ are themselves formal power series *beginning with* t^n:

$$\mathcal{P}_n(t) \sim p_{nn}t^n + \frac{p_{n+1,n}}{n+1}t^{n+1} + \frac{p_{n+2,n}}{(n+2)(n+1)}t^{n+2} + \cdots,$$

$$\mathcal{L}_n(t) \sim l_{nn}t^n + l_{n+1,n}t^{n+1} + l_{n+2,n}t^{n+2} + \cdots.$$

Definition. A sequence $\mathcal{S} : \{\mathcal{S}_n(t)\}$, $(n = 0, 1, \cdots)$ is a *triangular function set* if $\mathcal{S}_n(t)$ is a formal power series beginning with t^n or higher power; and the *characteristic numbers* of $\mathcal{S}$ are the coefficients of $1, t, t^2, \cdots$ in $\mathcal{S}_0(t)$, $\mathcal{S}_1(t)$, $\mathcal{S}_2(t), \cdots$ respectively

COROLLARY. *The characteristic numbers for $\mathcal{P}$ are the same as those for P, namely $\{p_n\}$.*

In addition to L we consider a second operator, $\mathcal{L}$, the *adjoint of* L:

$$(9) \qquad \mathcal{L} \equiv \mathcal{L}[u(t)] \equiv \sum_{n=0}^{\infty} \mathcal{L}_n(t)u^{(n)}(t).$$

THEOREM. *$\mathcal{L}$ carries triangular sets into triangular sets, and in particular, the identity $\mathcal{I} : \{\mathcal{I}_n(t) = t^n\}$ goes over into the triangular set $\mathcal{P} : \{\mathcal{P}_n(t)\}$:*

$$(10) \qquad \mathcal{L}[t^n] = \mathcal{P}_n(t).$$

The first part of the theorem is immediate. (10) can be proved in the way that (1) was established.

Consider now the equation

$$(11) \qquad \mathcal{L}[\mathcal{D}(t)] - \lambda\mathcal{D}(t) = 0.$$

THEOREM. *Equation* (11) *has a formal power series solution if and only if $\lambda = p_n$ $(n = 0, 1, \cdots)$; and if p_n is such that $p_i \neq p_n$, $(i \neq n)$, then there exists a unique formal series $\mathcal{D}(t)$* for $\lambda = p_n$, *and the coefficient of t^n in $\mathcal{D}(t)$ is not zero, but all preceding coefficients are zero.*

The result follows on substituting the series $\mathcal{D}(t) = d_0 + d_1t + \cdots$ into (11), and equating coefficients of like powers of t.

Definition. The triangular set $\mathcal{P}$ is complete if P is complete; and it is non-singular if P is non-singular.

THEOREM. *If $\mathcal{P}$ is complete there exists for each n one and only one power series $\mathcal{D}(t)$ whose first non-zero coefficient is that for t^n.*

Let $\mathcal{D}_n(t)$ be the unique power series corresponding to n. Then

(12) $$\mathfrak{L}[\mathfrak{D}_n(t)] - p_n\mathfrak{D}_n(t) = 0.$$

We term the power series $\{\mathfrak{D}_n(t)\}$ the *characteristic functions* for the operator $\mathfrak{L}$.

Since $\mathfrak{D}_n(t)$ begins with t^n (the coefficient being different from zero), we have the

COROLLARY. *The set $\mathfrak{D}$ of characteristic functions is non-singular.*

Definition. Let $\mathfrak{S} : \mathfrak{S}_n(t) \sim \sigma_{nn}t^n + \sigma_{n,n+1}t^{n+1} + \cdots$ be a triangular set. If $\mathfrak{T}$ is another such set, then $\mathfrak{S}\mathfrak{T}$ is the set

$$\mathfrak{S}\mathfrak{T} : (\mathfrak{S}\mathfrak{T})_n(t) \sim \sigma_{nn}\mathfrak{T}_n(t) + \sigma_{n,n+1}\mathfrak{T}_{n+1}(t) + \cdots.$$

THEOREM. *If $\mathfrak{P}$ is complete, $\mathfrak{D} : \{\mathfrak{D}_n(t)\}$ is a characteristic set for $\mathfrak{P}$; i. e.,*

(13) $$\mathfrak{D}\mathfrak{P} = \mathfrak{P}^*\mathfrak{D},$$

where

$$\mathfrak{P}^* : \mathfrak{P}_n^*(t) = p_n t^n.$$

Proof. For every set $\mathfrak{Q}$,

(14) $$\mathfrak{Q}\mathfrak{P} = \mathfrak{L}[\mathfrak{Q}].$$

Taking $\mathfrak{Q} = \mathfrak{D} : \mathfrak{D}\mathfrak{P} = \mathfrak{L}[\mathfrak{D}] = \mathfrak{P}^*\mathfrak{D}$.†

LEMMA. *If $\mathfrak{S}$ is non-singular then $\mathfrak{S}^{-1}$ exists such that*

$$\mathfrak{S}^{-1}\mathfrak{S} = \mathfrak{S}\mathfrak{S}^{-1} = \mathfrak{I} \quad (identity).$$

COROLLARY. *If $\mathfrak{P}$ is complete it has the canonical form*

(15) $$\mathfrak{P} = \mathfrak{D}^{-1}\mathfrak{P}^*\mathfrak{D}.$$

THEOREM. *If $\mathfrak{P}$ is complete, a necessary and sufficient condition that $\mathfrak{Q}$ be permutable with $\mathfrak{P}$ is that $\mathfrak{Q} = \mathfrak{D}^{-1}\mathfrak{Q}^*\mathfrak{D}$.*

4. *Associated Functional Equations; Formal Expansions.* With regard to L and $\mathfrak{L}$ we make the convention that L operates in the variable x and $\mathfrak{L}$ in t.

Let P be non-singular so that P^{-1} and L^{-1} exist.

THEOREM. *Formally we have* ‡

(17) $$L[P^{-1}(t;x)] \sim e^{tx} \sim \mathfrak{L}[P^{-1}(t;x)].$$

This follows from the relations

† $\mathfrak{P}^*\mathfrak{D}$ is the set $(\mathfrak{P}^*\mathfrak{D})_n(t) \sim p_n\mathfrak{D}_n(t)$.

‡ $P^{-1}(t;x)$ is *not* $\{P(t;x)\}^{-1}$.

$$L[P^{-1}(t;x)] \sim \sum_0^\infty (t^n/n!)L[P_n^{-1}(x)] \sim \sum_0^\infty t^n x^n/n!,$$

$$\mathcal{L}[P^{-1}(t;x)] \sim \sum_0^\infty (x^n/n!)\ \mathcal{L}[\mathcal{P}_n^{-1}(t)] \sim \sum_0^\infty x^n t^n/n!.$$

This result enables us to obtain a formal solution to the equations

(18) $\qquad L[y(x)] = f(x), \qquad$ (19) $\qquad \mathcal{L}[u(t)] = s(t).$

Let $f(x) = \sum_0^\infty f_n x^n$ be an entire function with $\limsup |f^{(n)}(0)|^{1/n} = \sigma$, so that $F(x) = \sum_0^\infty n!\, f_n x^n$ has a non-zero radius of convergence $(=1/\sigma)$. Then

$$f(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t}\, e^{x/t}dt,$$

C being a contour sufficiently close to, and surrounding, the origin.

THEOREM. (18) *has the formal solution*

$$y(x) \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t}\, P^{-1}\left(\frac{1}{t}\,;\, x\right)dt, \tag{20}$$

Γ *surrounding the origin.*

Proof. From (20),

$$L[y] \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t}\, L[P^{-1}\left(\frac{1}{t};\, x\right)]dt \sim \frac{1}{2\pi i}\int_\Gamma \frac{F(t)}{t}\, e^{x/t}dt = f(x).$$

Likewise we have the

THEOREM. (19) *has the formal solution*

$$u(t) \sim \frac{1}{2\pi i}\int_\Gamma \frac{S(x)}{x}\, P^{-1}\left(t;\, \frac{1}{x}\right)dx. \tag{21}$$

Formal solutions can be obtained in yet another way, namely in $\Theta_n(x)$-expansions and $\mathfrak{D}_n(t)$-expansions. Consider the equation with a parameter:

$$y(x) = f(x) + \lambda L[y(x)]. \tag{22}$$

The corresponding homogeneous equation is satisfied by $\Theta_n(x)$ for $\lambda = \lambda_n = 1/p_n$.

THEOREM. *If*

$$f(x) \sim \sum_0^\infty f_n \Theta_n(x) \tag{23}$$

then

(24) $$y(x) \sim \sum_0^\infty \lambda_n f_n \Theta_n(x)/(\lambda_n - \lambda)$$

is a formal solution of (22) *for every* $\lambda \neq \lambda_n$.

Proof. $L[y] \sim \sum_0^\infty f_n \Theta_n(x)/(\lambda_n - \lambda)$, so that both members of (22) agree. Likewise

THEOREM. *If*

(25) $$s(t) \sim \sum_0^\infty s_n \mathfrak{D}_n(t),$$

then

(26) $$u(t) \sim \sum_0^\infty \lambda_n s_n \mathfrak{D}_n(t)/(\lambda_n - \lambda)$$

is a formal solution of

(27) $$u(t) = s(t) + \lambda \mathcal{L}[u(t)]$$

for every $\lambda \neq \lambda_n$.

THEOREM. *The function* e^{tx} *has the expansion* *

(28) $$e^{tx} \sim \sum_{n=0}^\infty \Theta_n(x) \mathfrak{D}_n(t).$$

Proof. Set $e^{tx} \sim \sum_0^\infty \Theta_n(x) \mathcal{H}_n(t)$. From

(29) $$L[e^{tx}] = \mathcal{L}[e^{tx}]$$

we obtain

$$\sum_0^\infty p_n \Theta_n(x) \mathcal{H}_n(t) \sim \sum_0^\infty \Theta_n(x) \mathcal{L}[\mathcal{H}_n(t)],$$

so that

$$R(t;x) \sim \sum_0^\infty \{\mathcal{L}[\mathcal{H}_n(t)] - p_n \mathcal{H}_n(t)\} \Theta_n(x) \underset{t,x}{\equiv} 0.$$

Now let $M_{(k)}$ be an operator commutative with L and having the characteristic numbers

(a) $$\{\mu_n^{(k)}\}: \quad \mu_n^{(k)} = p_n, \quad n \neq k, \quad \mu_k^{(k)} \neq p_k.$$

Having the same characteristic polynomials $\Theta_n(x)$ as has L, then

$$(M - L)[R(t;x)] \sim \sum_{n=0}^\infty (\mu_n^{(k)} - p_n)\{\mathcal{L}[\mathcal{H}_n(t)] - p_n \mathcal{H}_n(t)\} \Theta_n(x) \underset{t,x}{\equiv} 0;$$

and on using (a) this reduces to

---

* Since $\Theta_n(x)$ and $\mathfrak{D}_n(t)$ are unique only to within arbitrary constant multipliers, it is assumed in (28) that these multipliers are suitably chosen. It suffices that the product of the coefficient of x^n in $\Theta_n(x)$ and the coefficient of t^n in $\mathfrak{D}_n(t)$ be $1/n!$.

$$\mathcal{L}[\mathfrak{H}_k(t)] - p_k\mathfrak{H}_k(t) = 0.$$

P being complete, $\mathfrak{H}_k(t)$ must then coincide * with $\mathfrak{D}_k(t)$; which establishes (28).

The expansion (28) is in terms of $\Theta_n(x)$, $\mathfrak{D}_n(t)$. Recalling how $P(t;x)$ is related to P, we see that e^{tx} bears the same relation to the identity set I. Now I has the characteristic numbers $\{i_n = 1\}$. This suggests an expansion for $P(t;x)$:

THEOREM. *$P(t;x)$ has the expansion*

$$P(t;x) \sim \sum_{n=0}^{\infty} p_n\Theta_n(x)\mathfrak{D}_n(t). \tag{30}$$

(30) follows from (7) and (28).

(30) may be regarded as a *canonical form* for $P(t;x)$. It puts in evidence the *characteristic polynomials* $\Theta_n(x)$, *functions* $\mathfrak{D}_n(t)$, and *numbers* p_n.

COROLLARY. *If P is complete, a necessary and sufficient condition that Q be permutable with P is that*

$$Q(t;x) \sim \sum_{0}^{\infty} q_n\Theta_n(x)\mathfrak{D}_n(t),$$

where $\{q_n\}$ are the characteristic numbers of Q.

If in (28) we expand in powers of t or of x we find:

$$x^n = \mathfrak{D}_0^{(n)}(0)\Theta_0(x) + \mathfrak{D}_1^{(n)}(0)\Theta_1(x) + \cdots + \mathfrak{D}_n^{(n)}(0)\Theta_n(x), \tag{31}$$

$$t^n = \Theta_n^{(n)}(0)\mathfrak{D}_n(t) + \Theta_{n+1}^{(n)}(0)\mathfrak{D}_{n+1}(t) + \Theta_{n+2}^{(n)}(0)\mathfrak{D}_{n+2}(t) + \cdots. \tag{32}$$

From these relations can be determined the formal expansions of functions of x or of t in $\Theta_n(x)$-series or $\mathfrak{D}_n(t)$-series respectively.

IV. SYSTEMS OF LINEAR EQUATIONS ASSOCIATED WITH SETS.

The theory of the operators L and $\mathcal{L}$ has a counterpart in the related field of systems of linear equations in infinitely many unknowns. This is the subject of the present section.

Consider again the operator

$$L:\quad L[y(x)] = \sum_{0}^{\infty} L_n(x)y^{(n)}(x), \tag{1}$$

where

$$L_n(x) = l_{n0} + l_{n1}x + \cdots + l_{nn}x^n.$$

* The other alternative: $\mathfrak{H}_k(t) \equiv 0$ is not possible, for it would mean that on the right hand side of (28) there was no term in $x^n t^n$.

On operating with L on the power series $y(x)=\sum_0^\infty y_n x^n$ we arrive at a system of linear expressions to which (1) is formally equivalent:

$$(2)\qquad L^*:\quad L^*\{y\}\equiv a_{ii}y_i + a_{i,i+1}y_{i+1} + a_{i,i+2}y_{i+2} + \cdots,\qquad (i=0,1,\cdots).$$

The quantities a_{ij}, l_{ij} are related by

$$(3)\qquad a_{nk} = k!\,[l_{k-n,0}/n! + l_{k-n+1,1}/(n-1)! + \cdots + l_{k,n}/0!],$$

$$(4)\qquad l_{kn} = (1/k!)\,[a_{nk} - \binom{k}{1} a_{n-1,k-1} + \cdots + (-1)^n a_{0,k-n}].$$

But the right hand member of (3) is seen to be precisely p_{kn}, so that

LEMMA. *a_{nk} is given by*

$$(5)\qquad a_{nk} = p_{kn}.$$

In other words, if we write out the infinite matrix representing the operator L^* of (2), then the elements in the successive columns are the coefficients of the polynomials $P_0(x)$, $P_1(x)$, $\cdots$.

The operator L^* carries vectors into vectors in space of infinitely many dimensions. In particular, if all components of a vector beyond the i-th are zero, the transformed vector has the same property. We term a sequence of vectors q^*: $q_n^* = (q_{n0}, q_{n1}, \cdots)$ a *set* if in q_n^* all components after the one of index n are zero. L^* carries sets into sets.

Let i^* be the identity set: $i_n^* = (0,\cdots,0,1,0,\cdots)$, the 1 appearing as the component of index n.

THEOREM. *L^* carries set i^* into set p^*: $L^*\{i_n^*\} = p_n^*$.*

With (2) we consider the system of homogeneous equations

$$(6)\qquad a_{ii}y_i + a_{i,i+1}y_{i+1} + \cdots = \lambda y_i,\qquad (i=0,1,\cdots).$$

THEOREM. *A vector solution of a finite number of components exists if and only if $\lambda = p_s$, $(s=0,1,\cdots)$; and if p_n is such that $p_i \neq p_n$, $i \neq n$, then to $\lambda = p_n$ corresponds only one vector solution with all components zero after that of index n (and this last component is not zero).*

THEOREM. *If $p_m \neq p_n$, $m \neq n$* for all *m, n, to each n exists one and only one solution of degree n; and the set of these solutions is the set* †

$$\theta^*:\ \theta_n^* = (\theta_{n0}, \theta_{n1}, \cdots, \theta_{nn}, 0, 0, \cdots):$$
$$L^*\{\theta_n^*\} = p_n\theta_n^*.$$

This is a consequence of the relation $(\Theta P)_n(x) = p_n\Theta_n(x)$.

Corresponding to $\mathfrak{L}$ there is a second system of linear expressions. In

† It is to be recalled that $\Theta_n(x) = \theta_{n0} + \theta_{n1}x + \cdots + \theta_{nn}x^n$ is the characteristic polynomial of degree n for P.

$$(7) \qquad \mathcal{L}: \mathcal{L}[u(t)] = \sum_{0}^{\infty} \mathcal{L}_n(t)u^{(n)}(t)$$

substitute $u(t) = \sum_{0}^{\infty} u_n t^n$. There results

$$(8) \qquad \mathcal{L}^*: \mathcal{L}^*\{u\} \equiv b_{i0}u_0 + b_{i1}u_1 + \cdots + b_{ii}u_i, \qquad (i = 0, 1, \cdots).$$

$\mathcal{L}^*$ carries a vector, whose first k components are zero, into another such vector. For this operator we define a *set* of vectors as a sequence $*q: \{*q_n\}$ in which all components of $*q_n$ of index less than n are zero. $\mathcal{L}^*$ carries sets into sets.

Lemma. *We have* $b_{ij} = j!p_{ij}/i!$.

Corollary. *In the matrix* $\| b_{ij} \|$ *which defines* $\mathcal{L}^*$, *the elements in the successive columns are precisely the coefficients of* $\mathcal{P}_0(t)$, $\mathcal{P}_1(t), \cdots$.

Theorem. $\mathcal{L}^*$ *carries the identity set* $*i$ *into* $*p$: $\mathcal{L}^*\{*i_n\} = *p_n$, *where the components of* $*p_n$ *are the coefficients of* $1, t, t^2, \cdots$ *in* $\mathcal{P}_n(t)$.

Theorem. *If* $p_m \neq p_n$, $m \neq n$, *the homogeneous equation*

$$\mathcal{L}^*\{u\} = \lambda u$$

has a solution if and only if $\lambda = p_n$, $(n = 0, 1, \cdots)$; *and to each* n *corresponds one solution* $*d_n$, *and its components of index less than* n *are zero. Moreover, the set* $*d: \{*d_n\}$ *of solutions is precisely the set*

$$*d_n = (0, \cdots, 0, d_{nn}, d_{n,n+1}, \cdots),$$

where

$$\mathcal{D}_n(t) \sim d_{nn}t^n + d_{n,n+1}t^{n+1} + \cdots.$$

Definition. If q^*, r^* are the sets

$$q_n^* = (q_{n0}, q_{n1}, \cdots, q_{nn}, 0, 0, \cdots), \quad r_n^* = (r_{n0}, \cdots, r_{nn}, 0, 0, \cdots),$$

then $(qr)^*$ is the set

$$(qr)_n^* = q_{n0}r_0^* + q_{n1}r_1^* + \cdots + q_{nn}r_n^*;$$

and if

$$*q: *q_n = (q_{nn}, q_{n,n+1}, \cdots), \qquad *r: *r_n = (r_{nn}, r_{n,n+1}, \cdots),$$

then

$$*(qr): *(qr)_n = q_{nn}*r_n + q_{n,n+1}*r_{n+1} + \cdots.$$

Theorem. *For* $p_m \neq p_n$, $m \neq n$, *the vector sets* p^*, $*p$ *have the canonical forms*

$$p^* = \theta^{*-1}(p^{**})\theta^*, \qquad *p = *d^{-1}(*p^*)d^*,$$

where† p^{**}, $*p^*$ *are the (same) diagonal sets*

$$p_n^{**} = *p_n^* = (0, 0, \cdots, 0, p_n, 0, \cdots).$$

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† The different notation p^{**}, $*p^*$ for the same vectors is to accord with previous notation.

CONCERNING SIMPLE CONTINUOUS CURVES AND RELATED POINT SETS.

By R. L. Wilder.

In his paper "Concerning Simple Continuous Curves," * R. L. Moore applied the term *simple continuous curve* to any point set which is either an arc, a simple closed curve, an open curve or a ray, and gave various topological characterizations for a point set that falls within the general class of such curves, as well as for the individual types of curves.†

The intent of the present paper is to supplement the work of Moore just cited, extending two of his results to more general spaces, and giving certain new definitions which the author believes may be of value.‡ As Moore made clear in his paper, the requirement of *boundedness* in a definition of arc may introduce great difficulties in certain problems. It will be noted that none of the definitions of arc, simple closed curve, etc., given in the present instance makes use of this condition. Furthermore, the condition that the set in question be *closed* is eliminated in several cases—a feature that would seem to be of importance in problems concerning non-closed sets.

To summarize briefly, before proceeding to the demonstrations: In § 1, Moore's characterization of simple continuous curves as a class is extended to any euclidean space, and is then applied to establish a new characterization based on a certain type of set which the author has found useful in other connections. In § 2 definitions are given of arc, the more general class of sets known as *irreducible connexes*,§ and of simple closed and *quasi-closed* curves.

In this connection it is indicated how a simple proof may be obtained of the theorem ¶ that every interval of an open curve is an arc; also a

* *Transactions of the American Mathematical Society*, Vol. 21 (1920), pp. 313-320. This paper will be referred to hereinafter as C. S. C.

† Other definitions have been given by various authors, but no attempt will be made here to give a complete bibliography of these. Cf. C. S. C., however.

‡ As indicated in following footnotes, the results of this paper were submitted to the American Mathematical Society several years ago, and as now published the details differ in some particulars from the original, due chiefly to the fact that by unification of treatment it is possible to present the various results in a single paper.

§ Cf. B. Knaster and C. Kuratowski, "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 206-255, §§ 2 and 3.

¶ Cf. R. L. Moore, "On the Fundations of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), pp. 131-164, Theorem 49, and C. S. C., Theorem 3. Both of the proofs given by Moore are for the plane.

simple proof of Moore's first definition of arc in C. S. C. is outlined, which establishes it for general spaces. In § 3 the notion of quasi-closed curve is used to obtain further definitions of simple closed curve, three of which require neither that the set in question be bounded nor that it be closed.

§ 1

*A property which characterizes simple continuous curves.**

It is well known † that if, in a plane S, C_1 and C_2 are two closed, mutually exclusive point sets and M is a bounded continuum having at least one point in common with each of the sets C_1, C_2, then there exists a point set H, a subset of M, such that H is connected and contains no point of either C_1 or C_2, but such that C_1 and C_2 each contains at least one limit point of H. In case C_1 and C_2 are subsets of M, we shall say that H, together with its limit points in C_1 and C_2, is a set $K(C_1, C_2)M$. In general, if M is not bounded, there may not exist any set $K(C_1, C_2)M$ for certain subsets C_1, C_2, of M, but it is easy to see that every continuum, whether bounded or not, contains some sets $K(C_1, C_2)M$ for the proper selections of C_1 and C_2.‡ We shall show that a continuum in which every set $K(C_1, C_2)M$ is an arc § must be a simple continuous curve.

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* The content of this section was presented to the American Mathematical Society April 2, 1926.

† Cf. Anna M. Mullikin, "Certain Theorems Relating to Plane Connected Point Sets," *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 144-162.

‡ For instance, if A and B are distinct points of M, let K_1 and K_2 denote circles with centers at A and radii d_1 and d_2, respectively, such that $d_1 < d_2 <$ distance AB. Let $K_i \cdot M = C_i$ $(i = 1, 2)$. Then M, whether bounded or not, contains a set $K(C_1, C_2)M$.

Since we shall not restrict ourselves to the plane in the present paper, it should be pointed out that Miss Mullikin's theorem cited above holds in euclidean space of any number of dimensions.

So far as the existence of sets $K(C_1, C_2)M$ is concerned, the condition of *connectedness im kleinen*, or *local connectedness*, as applied to connected sets in general, seems to imply more than *closed*. For in the case of a connected set M which is connected im kleinen, if C_1 and C_2 are mutually exclusive sets that are closed with respect to M, then M, whether bounded or not, contains a *bounded* set $K(C_1, C_2)M$. Cf. my paper "The Non-Existence of a Certain Type of Regular Point Set," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 439-446, Theorem 4. For other applications of the notion of a set $K(C_1, C_2)M$, or of related sets, see papers of mine in *Proceedings of the National Academy of Sciences*, Vol. 11 (1925), pp. 725-728, and Vol. 16 (1930), pp. 233-240.

§ As our basic definition of arc, we take the following: An *arc* is a closed, connected point set which is irreducibly connected between two points. Cf. N. J. Lennes, "Curves in Non-Metrical Analysis Situs with an Application in the Calculus

LEMMA 1. *A continuum M which has the property that every set $K(C_1, C_2)M$ is an arc is a continuous curve.*

Proof. Suppose there exists a continuum M such that every set $K(C_1, C_2)M$ is an arc and which is not a continuous curve. Then by a theorem due to R. L. Moore and the present author * there exist two concentric spheres K_1 and K_2 and a sequence of subcontinua of M, viz., M_∞, M_1, M_2, $M_3, \cdots$ such that (1) each of these continua contains at least one point of K_1 and K_2, respectively, but no point exterior to K_1 or interior to K_2, (2) no two of these sub-continua have a point in common, and no two of them contain points of any connected subset of M which lies wholly in the set $K_1 + K_2 + I$ (where I consists of all points whose distance from the common center of K_1 and K_2 is a number whose magnitude lies between the radii of K_1 and K_2 on the real number continuum), (3) M_∞ is the sequential limiting set of the sequence $M_1, M_2, M_3, \cdots$, (4) if K is that component

of Variations," *American Journal of Mathematics*, Vol. 33 (1911), pp. 285-326; G. H. Hallett, Jr., "Concerning the Definition of a Simple Continuous Arc," *Bulletin of the American Mathematical Society*, Vol. 25 (1919), pp. 325-326; B. Knaster and C. Kuratowski, *loc. cit.*, Theorem 27. The last two papers just referred to establish the fact that the word "bounded" may be omitted from Lennes' definition. That a set satisfying the above definition is compact, in very general spaces, may be shown by a method of argument similar to that used by R. L. Moore in the proof of Theorem 49 of his paper "On the Foundations of Plane Analysis Situs," *loc. cit.* (cf. footnote in Hallett's paper in this connection). From the argument of Knaster and Kuratowski, it is evident that such a set of points, when imbedded in a locally compact metric space, is homeomorphic with the linear interval [0, 1]. For their Lemma XXVI is true in such a space (the word "compact" being substituted for the word "bounded"), and the proof of their Theorem XXIII holds in such a space if one recalls a theorem of Alexandroff to the effect that a connected and locally compact metric space is separable. Cf. P. Alexandroff, "Über die Metrisation der im Kleinen kompakten topologischen Raume," *Mathematische Annalen*, Vol. 92 (1924), pp. 294-301. Accordingly, unless specifically stated otherwise, we shall assume throughout the present paper that the point sets considered are imbedded in such a space, although it will be clear that many of the proofs given below hold in more general spaces if an *arc* is defined as above for those spaces.

As several of the proofs given below depend upon the theorem that every two points of a continuous curve (i. e., a connected and connected im kleinen continuum) are the end-points of an arc of that curve, we point out that this theorem may be proved, on the basis of the above definition of arc, by a type of argument introduced by R. L. Moore, in "A Theorem Concerning Continuous Curves," *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

* Cf. R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377, Lemma 1. This lemma is true in any locally compact metric space whether the continuum in question is compact or not. Cf. G. T. Whyburn, *Bulletin of the American Mathematical Society*, Vol. 34, p. 409, abstract no. 18.

of $M \cdot (K_1 + K_2 + I)$ which contains M_∞, then all of the continua M_1, M_2, $M_3, \cdots$ lie in a single component U of $M - K$. If we let C_1 and C_2 be two distinct points of the set M_∞, it is clear that the set $U + C_1 + C_2$ is a set $K(C_1, C_2)M$ which is not an arc, and consequently the supposition that M is not a continuous curve leads to a contradiction.

The same proof evidently establishes the following lemma:

LEMMA 2. *If a continuum M has the property that every set $K(C_1, C_2)M$ is an arc, then every subcontinuum of M is a continuous curve.*

For the purposes of the present paper, we shall define an *end-point* of a continuous curve M to be a point of M that is not an interior point of any arc of M.*

LEMMA 3. *Let E be any set of end-points of a continuous curve M. Then $M - E$ is connected.*

For if $M - E$ were the sum of two mutually separated sets M_1 and M_2, an arc of M joining a point of M_1 to a point of M_2 would contain a point of E as an interior point.

LEMMA 4. *In any euclidean space, if no continuous subset of a continuum M has more than two boundary points with respect to M, then M is an arc, a simple closed curve, an open curve or a ray.*†

Proof. It is clear from the characterization of continua that are not continuous curves, quoted in the proof of Lemma 1 above, that M must be a continuous curve.

Suppose M contains a simple closed curve ‡ J. If $M - J$ is not vacuous,

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* Since this paper was originally written, it has been shown by G. T. Whyburn (for the plane) and W. L. Ayres (for any euclidean space and certain types of metric spaces), that an end-point in this sense is equivalent to an end-point as defined in my paper "Concerning Continuous Curves" (*loc. cit.*, p. 358). Cf. G. T. Whyburn, "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 369-400, Theorem 12, and W. L. Ayres, "Concerning Continuous Curves in Metric Space," *American Journal of Mathematics*, Vol. 51 (1929), pp. 577-594, Theorem 5.

† Lemma 4 is of course an extension of Moore's Theorem 6 of C. S. C., which was proved for the euclidean plane. It is important to observe that we are using the definition of boundary point just as given by Moore in this connection, viz., if M is a subset of N, the *boundary* of M with respect to N is the set of all points $[X]$ such that X is either a point or a limit point of M and also either a point or a limit point of $N - M$.

‡ By a simple closed curve we mean a sum of two arcs that have common end-points, but have no other points in common. By *open curve* and *ray* we denote point sets as defined by Moore, C. S. C., p. 341.

let P be a point of M that is not on J. Then M contains an arc PQ whose end-points are P and a point Q on J, and such that $PQ - Q$ is a subset of $M - J$. Let A be a point of $PQ - (P + Q)$, and let B and C be points of J neither of which is identical with Q. Let QB and QC denote arcs of J that do not contain C and B, respectively. Then it is clear that the set $AQ + QB + QC$, where AQ is that arc of PQ whose end-points are A and Q, is a continuous subset of M that has at least three boundary points with respect to M. Consequently, $M \equiv J$.

Suppose M contains no simple closed curve, and that it has at least two end-points, A and B. Then it is clear, from a method of argument similar to that used in the preceding paragraph, that M cannot contain any points other than those that lie on an arc of M from A to B.

Suppose M contains no simple closed curve and has one and only one end-point, A. Since every bounded continuous curve contains at least two points which do not cut it,* and since every non-cut point of an acyclic continuous curve is an end-point of that curve,† it is clear that M cannot be bounded. Then, by a theorem of Kuratowski,‡ A is the end-point of a ray in M. It is easy to see that M can contain no points that do not lie on this ray.

If M contains no simple closed curve and has no end-point, it follows as in the preceding paragraph that M is unbounded. Let P be any point of M. Then $M - P$ is the sum of two mutually separated sets, M_1 and M_2.§ Consider the continuous curve $M_1 + P$.¶ It follows easily from our hypothesis that P is a non-cut point of $M_1 + P$, and hence an end-point of $M_1 + P$. Consequently, as shown in the preceding paragraph, $M_1 + P$ contains a ray r_1

* Cf. S. Mazurkiewicz, "Un théorème sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 119-130.

† Cf. R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, Theorem 7. Although the sufficiency proof of this theorem (the part needed in the present connection) makes use of an accessibility theorem true in general only for the plane, the proof is easily modified so as to avoid this theorem. However, since the present paper was originally written, W. L. Ayres has shown (*loc. cit.*) that any point of a continuous curve M which is both a non-cut point of M and lies on no simple closed curve of M is an end-point of M. We have already noted above the validity of these theorems for end-points as defined in the present paper.

‡ C. Kuratowski, "Quelques propriétés topologiques de la demi-droite," *Fundamenta Mathematicae*, Vol. 3 (1922), pp. 59-64.

§ Cf. my paper "Concerning Continuous Curves," *loc. cit.*, and W. L. Ayres, *loc. cit.*

¶ That $M_1 + P$ is connected follows from a theorem of Knaster and Kuratowski (*loc. cit.*, Theorem VI).

with P as end-point. Similarly, $M_2 + P$ contains a ray r_2 with P as end-point. Evidently $r_1 + r_2$ is an open curve which is identical with M.

THEOREM 1. *In order that a continuum M in a euclidean space should be a simple continuous curve, it is necessary and sufficient that every set $K(C_1, C_2)M$ be an arc.*

Proof. The condition stated in the theorem is necessary. For let K denote some set $K(C_1, C_2)M$ of a simple continuous curve M, and let A and B be points of C_1 and C_2, respectively.

In case M is a simple closed curve, $M - (A + B)$ is the sum of two mutually separated sets, M_1 and M_2. Since $K - (A + B)$ is connected, it is a subset of M_1, say. But $M_1 + A + B$ is an arc and therefore a set which is irreducibly connected from A to B. Consequently $K \equiv M_1 + A + B$.

In case M is an arc, there exists only one arc, t, from A to B in M, and every connected subset of M which contains both A and B contains t. Hence (1) K contains t. On the other hand, since $M - (A + B)$ is the sum of the set $t - (A + B)$, and a set (vacuous or non-vacuous) which is separated from $t - (A + B)$, the subset $K - (A + B)$ of $M - (A + B)$ must lie wholly in $t - (A + B)$, a set with which (as already shown) it has points in common. Consequently, (2) t contains K. From (1) and (2) it follows that $t \equiv K$.

The proofs for the cases where M is a ray and an open curve are similar to the proof of the preceding paragraph.

The condition stated in the theorem is sufficient. For suppose M is not a simple continuous curve. Then it has a continuous subset, N, which has at least three distinct boundary points, A, B and C, with respect to M (Lemma 4). We note that by Lemma 2, N is a continuous curve.

The points A, B and C are end-points of N. Suppose C, for instance, is an interior point of an arc t of N, and let a and b be the end-points of t. Let R be a sphere with center at C and not enclosing a or b. Since, by Lemma 1, M is itself a continuous curve, there exists a sphere R_1 concentric with R such that all points of M interior to R_1 are joined to C by an arc of M which lies wholly interior to R. As C is a boundary point of N, there exists, interior to R_1, a point x of $M - N$. Let s be an arc of M with x and C as end-points and lying wholly interior to R. On s, in the order from x to C, let y be the first point of t. It is clear that y is distinct from a and b. Let that portion of s from x to y be denoted by u. Then the set $(u + t) - (a + b + x)$ is connected and $u + t$ is a set $K(a + b, x)M$. But $u + t$ is not an arc. Hence the supposition that C is not an end-point of N leads to a contradiction.

By Lemma 3, $N - (A + B + C)$ is connected. Hence N is a set

$K(A+B, C)M$ and is therefore an arc. This is absurd, since an arc is disconnected by the omission of any three distinct points. Hence the supposition that N has more than two boundary points with respect to M leads to a contradiction, and M is a simple continuous curve.

§ 2

*On connected sets which cut the plane.**

In a recent paper † I investigated some of the properties of a connected set M which contains more than one point and which remains connected on the omission of any connected subset. I found that M is a point set having properties very similar to those of a simple closed curve. Thus, if A and B are any two points of M, M is the sum of two sets K and N which are irreducibly connected from A to B and such that $K-(A+B)$ and $N-(A+B)$ are mutually separated. Furthermore, if M lies in a plane S, then S is cut by M in the sense that there exist at least two points, x and y, of $S-M$ which do not lie in any subcontinuum of $S-M$.

For the purposes of the present paper I shall call a point set which has the internal properties of the set M described in the last paragraph a *quasi-closed curve.*

LEMMA 5. *In order that a connected set M should be irreducibly connected between two of its points, A and B, it is necessary and sufficient that, if P be any point of M distinct from A and B, $M-P$ should be the sum of two mutually separated sets, K and N, neither of which contains both A and B.*

Proof. That the condition stated in this lemma is necessary follows at once from a theorem proved by Knaster and Kuratowski.‡

That the condition is sufficient is easily shown as follows: Suppose M is not irreducibly connected from A to B. Then it contains a proper connected subset, m, which contains both A and B. Let Q be a point of $M-m$. By hypothesis, $M-Q$ is the sum of two mutually separated sets, K and N, neither of which contains both A and B. As m is connected, and K and N are mutually separated, m is a subset of one of the latter sets, say K. But

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* The content of this section was presented to the American Mathematical Society, December 30, 1924.

† "On a Certain Type of Connected Set Which Cuts the Plane," *Proceedings of the International Mathematical Congress, Toronto, 1924*, University of Toronto Press, pp. 423-437.

‡ Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Theorem XVI.

m contains both A and B and therefore K contains both A and B contradictory to the hypothesis.

As a consequence of Lemma 5 an arc may be defined as follows:

THEOREM 2. *If A and B are distinct points, an arc from A to B is a closed and connected set of points M containing A and B such that if P is any point of M distinct from A and B, $M - P$ is the sum of two mutually separated sets neither of which contains both A and B.*

It is perhaps of interest to point out here that the definition of arc embodied in Theorem 2 is sufficient to prove Theorem 3 of C. S. C., to the effect that if A and B are two points of an open curve M the interval AB of M is an arc from A to B. The proof as given in C. S. C. shows that the interval AB satisfies the above definition. I mention this, since the proof of Def. 1 in C. S. C., upon which depends the proof of the fact that AB is an arc, is rather long and assumes the Zermelo Postulate.

It may be of interest, however, to give a simple proof of Moore's Def. 1, as well as to show that it holds in more general spaces than the euclidean plane:

THEOREM 3. *In a locally compact metric space, let A and B be distinct points, and M a closed and connected set containing A and B, such that (1) $M - A$ and $M - B$ are connected, (2) if P is any point of M distinct from A and B, then $M - P$ is the sum of two mutually separated connected sets. Then M is an arc from A to B.*

Proof. The set M is a continuous curve. For if not, there exist two concentric spheres K_1 and K_2, a sequence of sub-continua of M, viz., M_∞, M_1, M_2, $M_3, \cdots$, and sets K and U satisfying the conditions (1)–(4) outlined in the proof of Lemma 1 above. If we let $\bar{U}$ denote U together with its limit points, it is clear that $\bar{U}$ contains M_∞.

By hypothesis, all points of M_∞, except possibly two, disconnect M. But clearly none of these points disconnects $\bar{U}$. The set of all these points being non-denumerable, a violation is obtained of a theorem of R. L. Moore * to the effect that no continuum M contains a continuum $\bar{U}$ which contains a non-denumerable set of points that disconnect M but not $\bar{U}$. Thus the supposition that M is not a continuous curve leads to a contradiction.

* "Concerning the Cut-Points of Continuous Curves and of Other Closed and Connected Sets," *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106, Theorem B^*. Alexandroff has shown (*loc. cit.*) that every connected and locally compact space is perfectly separable, and hence satisfies the Lindelöf theorem, upon which the proof of Moore's theorem depends.

The curve M contains no simple closed curve, for if it does, another violation of the theorem of Moore just quoted results.

The curve M contains an arc from A to B. Denote this arc by t, and consider the set $M - t$. If $M - t$ is vacuous, the theorem is proved. If $M - t$ is non-vacuous, let C be one of its components, and let P be the boundary point of C in t (it is easy to see that C can have no other boundary point in t, since M contains no simple closed curve). It follows from condition (1) of the theorem that P is distinct from A and B. But then $M - P$ is the sum of *three* mutually separated sets, viz., C, C_A and C_B, where C_A is the component of $M - P$ that contains A, and C_B the set $M - (P + C + C_A)$. That C_B is not vacuous follows from the fact that it must contain the point B. But this is a violation of condition (2) of the theorem, and consequently $M - t$ is vacuous.

As another application of Lemma 5 we have the following theorem.

THEOREM 4. *In order that a connected set M should be a quasi-closed curve, it is necessary and sufficient that it contain no cut-points and be disconnected by the omission of any two of its points.*

Proof. That the conditions stated in the theorem are necessary follows from the properties stated in the first paragraph of this section, and the theorem of Knaster and Kuratowski referred to in the proof of Lemma 5.

The conditions stated in the theorem are sufficient. Let A and B be any two points of M. Then

$$M - (A + B) = K + N,$$

where K and N are mutually separated sets.

The set $M - A$ is connected by hypothesis. Hence $M - A$ is a connected set which is disconnected by the omission of a point B, and therefore $K + B$ and $N + B$ are connected. Similarly, $K + A$ and $N + A$ are connected. Let

$$k = K + A + B,$$
$$\text{and} \quad n = N + A + B.$$

Clearly k and n are connected sets.

Either K contains non-cut points of k or it does not. Suppose it does, and that P is such a point. Similarly, suppose that N contains a non-cut point, Q, of n. Then

$$M - (P + Q) = (k - P) + (n - Q).$$

That is, $M - (P + Q)$ is the sum of two connected sets which have a point,

A, in common and is therefore connected. This is a contradiction of the hypothesis. Hence either K contains no non-cut points of k or N contains no non-cut points of n. Suppose the latter to be the case. Then every point of n, except A and B, is a cut-point of n. Hence, if x is a point of N,

$$n - x = n_1 + n_2,$$

where n_1 and n_2 are mutually separated sets.

Neither of the sets n_1, n_2, contains both A and B. For suppose n_1 contains both A and B. Now

$$M - x = (K + n - x) = K + n_1 + n_2.$$

The sets K and n_2 are mutually separated, since n_2 is a subset of N and K and N are mutually separated. Hence $M - x$ is the sum of two mutually separated sets, $K + n_1$ and n_2, and M is disconnected by the omission of x, contrary to hypothesis. Thus neither of the sets n_1, n_2, contains both A and B.

Hence n is a connected set containing A and B such that if x is any point of n distinct from A and B, $n - x$ is the sum of two mutually separated sets neither of which contains both A and B. By Lemma 5, n is irreducibly connected from A to B.

Suppose K contains non-cut points of k, and let P be such a point. Let x be a point of N. Then

$$n - x = n_1 + n_2,$$

where n_1 and n_2 are connected sets such that n_1 contains A and n_2 contains B, and

$$M - P - x = (k - P) + n_1 + n_2.$$

But each of the connected sets n_1 and n_2 has a point in common with $k - P$, and since the latter set is connected, $M - P - x$ is connected contrary to hypothesis. Hence K cannot contain any non-cut points of k. That is, every point of k distinct from A and B disconnects k. That k is irreducibly connected from A to B can now be shown as in the case of n. It follows that M is a quasi-closed curve.

COROLLARY. *In order that a closed and connected set M should be a simple closed curve it is necessary and sufficient that M should contain no cut-points and be disconnected by the omission of any two of its points.**

* Since I presented this paper to the Society it has come to my attention that J. R. Kline has announced the result contained in this corollary for the case where the set M lies in euclidean space of two dimensions. Cf. his paper in *Proceedings of the*

§ 3

*On the definition of simple closed curve.**

LEMMA 6. *Let M be a set irreducibly connected from A to B, and let K be a subset of M which consists of all points of some arc, t, except an end-point, P, of that arc. Then P is a point of M.*

Proof. Suppose P is not a point of M. Let Q be the other end-point of t, and let x be a point of t distinct from P and Q, lying between † Q and B on M. Denote by m that portion of M from Q to B. The set m is irreducibly connected from Q to B.‡

The set $t-(P+Q)$ is a subset of m. For suppose y were a point of $t-(P+Q)$ lying between A and Q on M. Now $M-Q$ is the sum of two mutually separated connected sets, M_1 and M_2, neither of which contains both x and y.§ Let K_i denote the set of those points of $K-Q$ that lie in M_i $(i=1,2)$. Then $t-(P+Q)=K-Q=K_1+K_2$, and as K_1 and K_2 are mutually separated sets, $t-(P+Q)$ is not connected. As this is impossible, it follows that no points of $t-(P+Q)$ lie between A and Q on M.

The point B does not belong to t. For suppose it does. Then all of m belongs to t; for if y were a point of m not in t, it would follow from the fact that $m-y$ is the sum of two mutually separated sets containing Q and B, respectively, that $t-(P+Q)$ is not connected. Consider the set $t-B$. Since B cannot be either P or Q, $t-B$ is the sum of two mutually separated sets, t_1 and t_2, which contain P and Q, respectively. Since K is clearly identical with m, in view of what has been shown above, we must have $t_1 \equiv P$ and $t_2 \equiv m-B$. But then P is not a limit point of m, and consequently not of $t-P$. As this is impossible, the supposition that B is a point of t leads to a contradiction.

National Academy of Sciences, Vol. 9 (1923), pp. 7-12 (Theorem 4). Kline refers, in this connection, to an earlier result of Tietze, who stated a similar result but imposed the unnecessary condition of connectedness im kleinen on the set M, as well as certain restrictive conditions upon the space in which M is imbedded which do not hold when the space is an euclidean n-space $(n>1)$. Cf. *Mathematische Zeitschrift*, Vol. 5 (1919), p. 289. In the same connection Tietze has a result (for the same type of imbedding space) similar to Lemma 7 of the present paper.

* The content of this section was presented to the American Mathematical Society January 1 and September 8, 1926.

† We are of course referring here to the linear order on M. Cf. Knaster and Kuratowski, *loc. cit.*, Theorem XX.

‡ Knaster and Kuratowski, *loc. cit.*, Corollary XXIV.

§ Cf. Knaster and Kuratowski, *loc. cit.*, pp. 219-220.

Denote the set $t - P$ by m_1 and the set $m - m_1$ by m_2. As m_2 contains B, m_2 is non-vacuous. Since we have supposed P not a point of M, m_2 can contain no limit point of m_1. Then since m is connected, m_1 contains a limit point, C, of m_2. Let D be a point of t between C and P. It is easy to see that D is between C and B on m. Denote the portion of t from Q to C by QC. The set m_2 being connected,* it follows that $QC + M_2$ is a connected set containing Q and B. But this set does not contain D, and hence is a proper connected subset of m containing Q and B, contradicting the fact that m is irreducibly connected from Q to B. Thus the supposition that P is not a point of M leads to a contradiction.

THEOREM 5. *Let M be a connected im kleinen continuum which is the sum of two sets N_1 and N_2, each irreducibly connected from A to B, such that $N_1 \cdot N_2 = A + B$. Then M is a simple closed curve.*

Proof. Let

$$N_1 - (A + B) = n_1,$$
$$N_2 - (A + B) = n_2.$$

If n_1 and n_2 are mutually separated, it follows at once that N_1 and N_2 are closed sets and therefore arcs with end-points A and B. In this case M is a simple closed curve.

Suppose that n_1 and n_2 are not mutually separated, and that, for the sake of definiteness, n_2 contains a limit point, P, of n_1. I shall show that in this case M cannot be connected im kleinen.

The point P is a sequential limit point of a set of distinct points, $P_1, P_2, P_3, \cdots$ all belonging to n_1. There exists, between the points of n_1 and the points of the linear continuum $0 \leq x \leq 1$, a one-to-one correspondence T in which order is preserved. For every positive integer n, let the point of the linear continuum corresponding to P_n under the correspondence T be denoted by x_n; suppose the point whose abscissa is zero corresponds to A, and is denoted by a, and that the point whose abscissa is 1 corresponds to B and is denoted by b. The set of points $\{x_n\}$ has at least one limit point, x, and x is a limit point of some subset of this set which is contained, say, in the interval ax. Call this subset X. Then x is a sequential limit point of a sequence of distinct points of X, viz., $a_1, a_2, a_3, \cdots$, where for every positive integer $n > 1$, a_n lies on the interval $a_{n-1}x$.

For every positive integer n, let the point of the sequence $P_1, P_2, P_3, \cdots$

* Knaster and Kuratowski, *loc. cit.*, Theorem XV.

which corresponds to a_n under the correspondence T be denoted by y_n. Clearly P is a sequential limit point of the sequence $y_1, y_2, y_3, \cdots$. Let that point of N_1 which corresponds to x under the correspondence T be denoted by Y.

The point Y is a limit point of that portion of N_1 from A to Y, and there exists a sequence of distinct points of N_1, viz., $z_1, z_2, z_3, \cdots$ having Y as a sequential limit point, and such that for every positive integer $n \geqq 1$, the point z_n lies between y_n and y_{n+1} on N_1. Let that portion of N_1 from y_n to z_n be denoted by t_n. The limit set * of the sequence of sets $t_1, t_2, t_3, \cdots$ is a closed and connected † point set t, containing P and Y.

Let K be a spherical neighborhood with center P and such that Y is exterior to K. Denote the point set consisting of K and its frontier, F, by k. There exists a continuum N which is a subset of t and of k, and contains P and at least one point of F.‡ The set N contains no points of N_1. For suppose Q is a point of N_1 belonging to N. Then Q is a point of that portion of N_1 from Y to B, and, since it is a point of t, a limit point of that portion of N_1 from A to Y. The point Q is therefore identical with Y. But this is absurd, since Y is not a point of k. Hence N contains no points of N_1.

As M is closed, N is a subset of M, and therefore of n_2. Then N is an arc, since every closed and connected subset of an irreducible connexe is an arc.§ Let C be a point of this arc distinct from its end-points. There exists a spherical neighborhood T_1 with center at C such that if t_1 denotes the point set consisting of T_1, plus its frontier, then t_1 contains neither Y, A, B, nor any points of N_2 that are not also points of N. As M is a continuous curve, there exists, concentric with and lying interior to T_1, a spherical neighborhood T_2, such that if a and b are points of M lying interior to T_2, there exists an arc ab whose end-points are a and b, is a subset of M, and lies wholly interior to T_1.

There exists a positive integer j such that y_j and y_{j+1} lie interior to T_2 and z_j lies exterior to T_1. There exists an arc s which is a subset of M, has y_j and y_{j+1} as end-points, and lies wholly interior to T_1. No points of s belong to N_2. For suppose such points exist. Call the set of such points r. All points of r are obviously points of N, and r is closed, being the set of points common to two closed sets. Hence, as y_j is not a point of r, there

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* By the limit set of a sequence of sets $M_1, M_2, M_3, \cdots$ is meant the set of all points $\{x\}$, such that x is a limit point of some set of points $m_1, m_2, m_3, \cdots$ where for every positive integer n, m_n is a point of M_n.

† Cf. S. Janiszewski, "Sur les continus irreductibles entre deux points," *Journal de L'Ecole Polytechnique* (2), Vol. 16 (1912), p. 98, Theorem 1.

‡ Cf. Anna M. Mullikin, *loc. cit.*, Theorem 1.

§ Cf. Knaster and Kuratowski, *loc. cit.*, Corollary XXVIII.

exists on s, in the order from y_j to y_{j+1}, a first point, D of r, and D is distinct from y_j. That portion of s from y_j to D is an arc e. All points of e except D belong to N_1. Hence D is a point of N_1, by the above Lemma 6. But as D is obviously distinct from A and B, it must be a point common to n_1 and n_2. This is a contradiction of the hypothesis. Hence the supposition that s contains points of N_2 leads to a contradiction. It follows that s is a subset of N_1, and is identical with that portion of N_1 from y_j to y_{j+1}. But as z_j is a point of this portion, z_j is therefore a point of s. This is impossible, as s contains no points exterior to T_1.

Thus the supposition that n_1 and n_2 are not mutually separated leads to a contradiction, and the theorem is proved.

LEMMA 7. *If M is a connected im kleinen set which is irreducibly connected from A to B, then M is a simple continuous arc having A and B as end-points.**

Proof. It is necessary to prove only that M is closed.

Suppose M is not closed. Then there exists a point P which is a limit point of M and does not belong to M. There exists a correspondence, T, preserving order, between the points of M and the set of points of the linear continuum $0 \leq x \leq 1$, in which A and B correspond to the points whose abscissas are 0 and 1, respectively. Denote the latter points by a and b, respectively. As in the proof of Theorem 5, it can be shown that there exist, on M, two sequences of points, $y_1, y_2, y_3, \cdots$ and $z_1, z_2, z_3, \cdots$, such that (1) the set of all points on ab corresponding, under the correspondence T, to points of these sequences has a sequential limit point, x, whose transform in M is a point Y, (2) all points of these sequences lie between A and Y (or Y and B) on M, and for every positive integer $n \geq 1$, z_n lies between y_n and y_{n+1} on M, (3) Y is the sequential limit point of the sequence $z_1, z_2, z_3, \cdots$, and P is the sequential limit point of the sequence $y_1, y_2, y_3, \cdots$.

That M is not connected im kleinen at Y is shown very easily by considering a spherical neighborhood T with center Y and such that P is exterior to T.

THEOREM 6. *If M is a quasi-closed curve which is connected im kleinen, then M is a simple closed curve.*

* Since this paper was originally written, the result stated in this lemma has been given with a different proof by G. T. Whyburn, in his paper "Concerning Regular and Connected Point Sets," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 685-689; also, a different proof of the lemma has been given by the present author in his paper "On Connected and Regular Point Sets," *ibid.*, Vol. 34 (1928), pp. 649-655.

Proof. As M is a quasi-closed curve, it is the sum of two sets M_1 and M_2 which are irreducibly connected between two points, A and B, of M, and such that $M_1 - (A + B)$ and $M_2 - (A + B)$ are mutually separated. That M_1, say is connected im kleinen at all points distinct from A and B is evident. To show that it is connected im kleinen at the latter points, consider the point A in particular. If M_1 is not connected im kleinen at A, there exists a spherical neighborhood K_1 with center at A, not enclosing B, such that if K_2 is any neighborhood concentric with K_1 and lying interior to K_1, then K_2 encloses a point x of M_1 such that the portion of M_1 from x to A contains points exterior to K_1. Since M is connected im kleinen at A, there exists a neighborhood C with center at A, such that if P is any point of M interior to C, P is joined to A by a connected subset of M which contains no points of M exterior to K_1. Now there exists a point x of M_1 interior to C, such that the portion of M_1 from x to A contains points exterior to K_1. But there does exist a connected subset, N of M, containing both x and A and lying interior to K_1. Obviously N is not a subset of M_1. Let

$$N \cdot (M_i - A) = N_i, \qquad (i = 1, 2).$$

Then $N - A$ is the sum of two sets N_1 and N_2, where N_1 and N_2 are mutually separated sets. The set $N_1 + A$ is a connected subset* of M_1 containing x and A and lying wholly interior to K_1, which is clearly impossible. Thus M_1 is connected im kleinen at A and similarly at B. Likewise M_2 is connected im kleinen at both A and B. The theorem now follows as a consequence of Lemma 7.

Lemma 8. *Let M be a connected point set such that if A and B are any two distinct points of M, then $M - (A + B)$ is the sum of two mutually separated connected sets. Then M is a quasi-closed curve.*

Proof. The set M contains no cut-points. For, suppose there exists in M a point A such that $M - A$ is the sum of two mutually separated sets, K and N. Then K and N are connected. For if K, say, is the sum of two mutually separated sets K_1 and K_2, then, if B is a point of N,

$$M - (A + B) = K_1 + K_2 + (N - B),$$

and the sets K_1, K_2 and $N - B$ are mutually separated, thus contradicting the condition of the theorem.

* Knaster and Kuratowski, *loc. cit.*, Theorem VI.

The sets K and N contain no cut-points. For if K, say, contains a point B such that $K-B$ is the sum of two mutually separated sets K_1 and K_2,

$$M-(A+B)=K_1+K_2+N,$$

where K_1, K_2 and N are mutually separated sets, thus contradicting the condition stated in the theorem.

Then, if P is a point of K, and Q is a point of N, $K-P$ and $N-Q$ are connected sets which have a common limit point, A. Hence

$$M-(P+Q)=(K-P)+(N-Q)+A,$$

and therefore $M-(P+Q)$ is the sum of two connected sets $(K-P)+A$ and $(N-Q)+A$ having in common the point A, and is therefore connected, contradicting the condition stated in the theorem.

Thus the supposition that M contains a cut-point leads to a contradiction, and the fact that M is a quasi-closed curve follows from Theorem 4.

THEOREM 7. *In order that a continuum M should be a simple closed curve, it is necessary and sufficient that if A and B are any two distinct points of M, then $M-(A+B)$ is the sum of two mutually separated connected sets.*

This theorem is a consequence of Lemma 8.

THEOREM 8. *In order that a connected and connected im kleinen set M should be a simple closed curve, it is necessary and sufficient that if A and B are any two distinct points of M, then $M-(A+B)$ is the sum of two mutually separated connected sets.*

Theorem 8 is a consequence of Lemma 8 and Theorem 6. It will be noted that the definition of simple closed curve embodied herein does not require that the set M be either closed or bounded. This is also characteristic of the following two definitions.

THEOREM 9. *In order that a point set M should be a simple closed curve, it is necessary and sufficient that it be connected and connected im kleinen, and that it should contain no cut-points and be disconnected by the omission of any two of its points.*

Theorem 9 is a consequence of Theorems 4 and 6.

THEOREM 10. *In order that a connected and connected im kleinen point*

set M should be a simple closed curve, it is necessary and sufficient that it remain connected upon the omission of any connected subset.

Theorem 10 is a consequence of the result referred to at the beginning of § 2, and of Theorem 6.

Theorem 10 is perhaps more striking than any of the others, in view of the fact that there exist * connected and connected im kleinen point sets which contain no continua whatsoever and in that the addition of the simple condition that the set remain connected upon the omission of any connected subset is sufficient to render the set into such a simple continuum as a simple closed curve.

It seems to me probable that the following theorem is true: *Let M be a connected im kleinen point set which is the sum of two sets M_1 and M_2 which are irreducibly connected between two points A and B and which have only A and B in common. Then M is a simple closed curve.* I have not yet been able to establish it, if true, but it obviously includes Theorems 5 and 6. and is more general than either.

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* Cf. R. L. Wilder, "A Connected and Regular Point Set Which Has No Subcontinuum," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 332-340; also B. Knaster and C. Kuratowski, "A Connected and Connected im Kleinen Point Set Which Contains No Perfect Subset," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 106-109.

NOTE ON FUNCTIONS OF R-TH DIVISORS.

By E. T. Bell.

1. The term r-th divisors of n (r, n integers, $r \geqq 0$, $n > 0$) is used by D. H. Lehmer * to designate the set of positive integers defined as follows: $\delta_0 = n$; $\delta_r \equiv$ the set of all divisors of all the integers in the set δ_{r-1} ($r \geqq 1$). The set d_r of r-th proper divisors is defined similarly with the restriction that a proper divisor of an integer $m > 0$ is a divisor $< m$ of m. If r is not an integer $\geqq 0$, the definition can easily be extended, and in fact r may be any real or complex number provided that u^r, which is defined presently, is interpreted in the irregular field IF of all numerical functions as demanded by the postulates of the field; namely as $\exp(r \log u)$. The theory of transcendental analytic functions in IF is developed in a paper to appear shortly.†

2. The entire theory of functions of the numbers δ_r, d_r is implicit in the equations in IF.

2.1 $$_rf = u^r f, \qquad _rf' = (u-1)^r f,$$

in which f is an arbitrary numerical function (or arbitrary element of IF), u is defined by $u(n) = 1$ for all integers $n > 0$, and the definitions of $_rf$, $_rf'$ are (for all integers $n > 0$), $_rf(n) \equiv \sum f(\delta_r)$, $_rf'(n) \equiv \sum f(d_r)$, $_0f = {}_0f' = f$. The notation in 2.1 is as in any of the papers cited in § 5; the $\sum$'s refer to all numbers of the respective sets δ_r, d_r. To prove 2.1, it is sufficient to translate the definitions of $_rf(n)$, $_rf'(n)$ into the notation of IF; namely

$$_rf = u_{r-1}f, \qquad _rf' = (u-1)_{r-1}f',$$

for all integers $r > 0$. As in IF, the equations 2.1 then define $_rf$, $_rf'$ for all real or complex numbers r.

Since u is regular in IF it has a unique inverse, $u^{-1} = \mu$, (Möbius' or Mertens function). To develop the properties of $_rf$, $_rf'$ ad libitum it is sufficient merely to manipulate the right-hand members of 2.1 according to the rules of elementary algebra, not dividing by f if f is irregular (f is irregular in IF if and only if $f(1) = 0$), as IF is an instance of an abstract irregular field. If at any stage it be required to interpret the results in

* *American Journal of Mathematics*, Vol. 52 (1930), pp. 293-304.

† *Transactions of the American Mathematical Society*. Further references are collected in Section 5.

terms of ${}_rf$, ${}_rf'$, this may be done (for all r) by the substitutions, equivalent to 2.1),

2.2 $$f = {}_rf/u^r = \mu^r {}_rf = {}_rf'/(u-1)^r.$$

3. A few examples will suffice to show how the properties of ${}_rf$, ${}_rf'$ follow by elementary manipulations of 2.1. First may be noted the alternative definitions for all integers r: u^rf is the r th so-called numerical integral, and $(u-1)^rf$ the r-th proper numerical integral of f. If ${}_rf$, ${}_rf'$ be regarded as having been obtained by this process, instead of by summations over δ_r, d_r, we shall write f_r, f_r'. Thus

3.1 $${}_rf = f_r = u^rf, \qquad {}_rf' = f_r' = (u-1)^rf,$$

and as in 2.1 we take for all real or complex numbers r as definitions of f_r, f_r' the elements u^rf, $(u-1)^rf$ of IF. In passing it may be repeated (as pointed out in the papers cited in Section 5) that the numerical integral uf of Bugaieff is in no way distinguished from any other simple (binary) product in IF, and similarly for the Liouville-Dedekind inversion; namely, if $uf = F$, then $f = F/u = \mu F$, which is on the same footing as $gf = h$, $f = h/g = g^{-1}h$, where g is *any* regular element of IF and g^{-1} is its reciprocal in IF .

Let r be an integer. If in IF a product gf is factorable(see Section 5, references), and one of its factors, say g, is factorable, then f is factorable. Since u^r is factorable, a necessary and sufficient condition that $f_r(={}_rf)$ be factorable is that f be factorable. Hence if for any particular integer r, f_r is factorable, it is factorable for all integers r.

As a second example, since $u^{r+s} = u^ru^s = u^su^r$, and similarly for $u-1$, we see from 2.1 that

$${}_{r+s}h = h_{r+s} = {}_rh_s = {}_sh_r \qquad (h = f \text{ or } f').$$

Again, if $F = uf$, and it be required to express the inversion formula $f = F/u$ in terms of functions F_r, we write $u-1 = v$ for convenience, so that

$$f = F/u = F/(1+v) = F(1 - v + v^2 - \cdots) = \sum_{j=0} (-1)^j F_j.$$

The last series obviously terminates when f has the argument n (an arbitrary finite integer), as $F_j(n) = 0$ for $j =$ a certain finite integer, from the definition of d_j.

The connection between ${}_rf$, ${}_rf'$ is obvious from 2.1. For simplicity let r be an integer ≥ 0. Then, $(v \equiv u-1)$,

$$(-1)^r {}_r f' = (1-u)^r f = \sum_{j=0}^{r} (-1)^j u^j f = \Sigma\, (-1)^j \tbinom{r}{j}\, {}_j f;$$

$$ {}_r f = u^r f = (1+v)^r f = \sum_{j=0}^{r} \tbinom{r}{j} v^j f = \Sigma\, \tbinom{r}{j}\, {}_j f'.$$

As another identity suggested by 2. 1 we have

$$f + (-1)^r {}_r f = f\,[1-(1-u)^r] = \sum_{j=1}^{r} (-1)^{j-1} \tbinom{r}{j} u^{j-1} uf,$$

and hence, if $F = uf$,

$$f + (-1)^r {}_r f = \sum_{j=1}^{r} (-1)^{j-1} \tbinom{r}{j} F_{j-1}.$$

An alternative form has F_i instead of ${}_i F$, since

$$u^{j-1} uf = u^{j-1} F = {}_{j-1} F = F_{j-1}.$$

The binomial theorem for a positive integral exponent r has a variety of interpretations. To take only one, $u - v = 1$; hence

$$f = (u-v)^r f = \sum_{j=0}^{r} (-1)^j \tbinom{r}{j} u^j v^{r-j} f,$$

and we write $u^j v^{r-j} f$ in either of the forms (among others) $u^j (v^{r-j} f)$, $v^{r-j}(u^j f)$, and hence as any one of

$$u^j\, {}_{r-j} f', \quad u^j f'_{r-j}, \quad v^{r-j}\, {}_j f, \quad v^{r-j} f_j,$$

that is as any one of the eight

$$({}_{r-j} f')_j, \quad {}_j({}_{r-j} f'), \quad (f'_{r-j})_j, \quad {}_j(f'_{r-j}),$$
$$({}_j f')_{r-j}, \quad {}_{r-j}({}_j f)', \quad (f_j)'_{r-j}, \quad {}_{r-j}(f_j)'.$$

Hence f has (among many more) the eight binomial expressions

$$f = \sum_{j=0}^{r} (-1)^j \tbinom{r}{j} F^{(j)},$$

where $F^{(j)}$ denotes any one of the above functions. The forms of the $F^{(j)}(n)$ are immediately written out from the notation: a prefix s denotes a sum over s-th divisors, a suffix s an s-iterated numerical integration, an accent indicates proper, lack of an accent all divisors of the set, and operations within the ( ) are to be performed before those outside. This example illustrates the commutative, associative and distributive laws in IF, which are abstractly identical with the like in a field.

Finally, by the simple properties of what were called functional powers in *IF*, all formulas for f_r, ${}_rf'$, $\cdots$ can be transposed from sum to product forms without computations. In the papers cited in Section 5 the extension to functions of elements in any commutative semigroup having a unique-decomposition theorem (as for example in any finite abelian group) was noted and developed.

4. The explicit forms of ${}_rf$, ${}_rf'$ (and hence of f_r, f_r') are written down at once from the generator (see papers 5.3 and that of 1915) of f. If f is factorable, and $F(x, z)$ is its generator, the generator of ${}_rf$ is $F(x,z)/(1-z)^r$. If $h_a(x)$ is the coefficient of z^a in the formal development of this generator as a power series in z, the explicit form of $f_r(n)$ is

4.1 $$f_r(n) = \prod h_a(p),$$

where $n = \prod p^a$ is the prime decomposition of n. If f is not factorable, $u^r f$ is the essentially simplest form of ${}_rf$; the form corresponding to 4.1 can be obtained as in the papers cited. To find the explicit form of ${}_rf'$ we apply to the explicit form of f_r the theorem $(-1)^r {}_rf' = \sum (-1)^j \binom{r}{j} {}_jf$.

The simple, general method for obtaining the generator of a given f from its arithmetical definition was given in the previous papers, and in that of 1915 a list of the generators of practically all of the factorable numerical functions in the literature, with many more, was written out. Some of the simplest are: the generator of u is $1/(1-z)$; that of Euler's ϕ is $(1-z)/(1-xz)$; that of σ_k (sum of k-th powers of all divisors) is $1/(1-z)(1-x^k z)$; that of μ is $1-z$; that of the unit function η in *IF* is 1; that of the zero function ω in *IF* is 0.

The definitions of ${}_rf$, ${}_rf'$ connect an arbitrary numerical function with the very special ones u, $u-1$. If for the latter any elements g, h, $\cdots$ of *IF* be substituted, the treatment is precisely the same. From the standpoint of *IF*, which is the irregular field of *all* numerical functions, there is no particular reason for distinguishing u, $u-1$, $\cdots$ from any other elements of *IF*.

5. A concise account of the theory is given in *Algebraic Arithmetic,* (*American Mathematical Society Colloquium Publications No. 7,* 1927), and an adequate summary in *The Journal of the Indian Mathematical Society,* Vol. 17 (1928). It was however developed first in *University of Washington Publications in Science,* Vol. 1, No. 1 (1915), pp. 1-44, where there are numerous applications. For the specific parts used in this paper we may refer to

5. 1 *Tohoku Mathematical Journal,* Vol. 17 (1920), pp. 221-231.

5. 2 *Bulletin of the American Mathematical Society,* Vol. 28 (1922), pp. 111-122.

5. 3 *Transactions of the American Mathematical Society,* Vol. 25 (1923), pp. 135-154.

5. 4 *L'Enseignement Mathématique,* t. 23 (1923), pp. 305-308.

The last contains everything necessary for the present paper except the theory of generators, which will be found in 5. 3.

It may be pointed out that the generalizations to any systems having unique-decomposition theorems are not true generalizations in the sense of modern algebra; they are merely other solutions of a certain set of postulates. To obtain true generalizations, the set of postulates must be modified in the direction of weakness.

For the theory of irregular fields, see either *Algebraic Arithmetic* above, or

5. 5 *Annals of Mathematics,* Vol. 27 (1926), pp. 511-536.

THE UNIFORM APPROXIMATION OF A SEQUENCE OF INTEGRALS.

By R. L. JEFFERY.

Let $g_1, g_2, \cdots, g_p$ be p functions of x summable on (a, b). It has been shown by Lebesgue.* that there exists a finite sub-division α of (a, b), and on each interval of this sub-division with length α_i a point ξ_i of x such that

$$(1) \qquad \left| \int_a^b g_h dx - \sum_i g_h(\xi_i)\alpha_i \right| < \epsilon \quad (h = 1, 2, \cdots, p).$$

Lebesgue remarks † that such an approximation can evidently be made for an infinite sequence of functions. Our investigation seems to show that this remark needs some qualification, even when the sequence is bounded in x and n. In Lebesgue's results, for a given ϵ, both α_i and ξ_i are fixed. We show that it is always possible to find ξ_i on each interval with length α_i so that (1) holds, for any α with norm sufficiently small. Conditions are also determined under which the approximation can be made, independent of n, for an infinite sequence of functions. The paper concludes with some applications to functions of two variables.

It will make for brevity if we agree once for all that α shall represent a finite sub-division of (a, b), and β a sub-set of the intervals of α; α_i and ξ_i, β_j and ξ_j shall denote the length of and a point on an interval of α and β respectively.

THEOREM I. *Let $g_1, g_2, \cdots, g_p$ be p functions of x summable on the measurable set E contained on (a, b). Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to choose from α with norm sufficiently small a sub-set of intervals β, and on each interval of β a point ξ_j of E, such that*

$$\left| \int_E g_h dx - \sum_j g_h(\xi_j)\beta_j \right| < \epsilon \quad (h = 1, 2, \cdots, p),$$

and at the same time $| mE - m\beta | < \epsilon'$. If E is the interval (a, b) then the approximating sums can be taken over all the intervals of α.

We shall first establish the theorem for two bounded functions, g_1, and g_2. Let l and L be the bounds of the set defined by g_h $(h = 1, 2)$. Divide (l, L)

* *Annales de Toulouse*, (3), Vol. I, p. 33.

† *loc. cit.*, p. 34.

into n equal parts each of length η, where n is large enough to insure the following results:

(a) $$\eta(b-a) < \epsilon/3.$$

(b) $$\left| \int_E g_h dx - \sum_i \{l+(i-1)\eta\} me_i^h \right| < \epsilon/3 \qquad (h=1,2),$$

where e_i^h has the usual significance. Let e_{ij} be the set of points common to e_i^1 and e_j^2. Let t be greater than zero but otherwise arbitrary. Following the method used by Lebesgue, we can associate with each set e_{ij} a finite set of non-overlapping intervals a_{ij} with the following properties:

(1) Each interval of a_{ij} contains at least one point of e_{ij}, and the measure of the part of e_{ij} not on a_{ij} is less than t.

(2) The measure of the part of e_{ij} on a_{ij} differs from the measure of a_{ij} by not more than t.

(3) There are no points common to any of the n^2 sets a_{ij} $(i, j = 1, 2, \cdots, n)$.

(4) The measure of a_{ij} and e_{ij} differ by not more than t. This follows from (1) and (2).

If U is the larger of the two numbers $|L|$ and $|l|$, and

$$S_1^h = \sum_i \{l+(i-1)\eta\} me_i^h \qquad (h=1,2),$$
$$S_2^1 = \sum_{i=1}^n \{l+(i-1)\eta\} \sum_{j=1}^n ma_{ij},$$
$$S_2^2 = \sum_{j=1}^n \{l+(i-1)\eta\} \sum_{i=1}^n ma_{ij},$$

then from (4) and the fact that

$$e_i^1 = \sum_{j=1}^n e_{ij}, \quad \text{and} \quad e_j^2 = \sum_{i=1}^n e_{ij},$$

we get

(5) $$|S_1^h - S_2^h| < n^2 U t \qquad (h=1,2).$$

Let M be the largest number of intervals in any of the n^2 sets a_{ij}. Let $\delta > 0$ be less than the length of the smallest interval in any of these n^2 sets, but otherwise arbitrary. Let α be any finite sub-division of (a, b) for which $\alpha_i < \delta$, and let u_{ij} be the intervals of α which have a part in common with a_{ij}, and which contain points of e_{ij}. Then mu_{ij} can neither be greater than ma_{ij} by more than $2M\delta$, nor, on account of (2), can it be less than ma_{ij} by more than t. Hence in any case we have

(6) $$| mu_{ij} - ma_{ij} | < 2M\delta + t.$$

Consequently, if we write

$$S_3^1 = \sum_{i=1}^{n} \{l + (i-1)\eta\} \sum_{j=1}^{n} mu_{ij},$$

and

$$S_3^2 = \sum_{j=1}^{n} \{l + (j-1)\eta\} \sum_{i=1}^{n} mu_{ij},$$

it is easily seen that

(7) $$| S_2^h - S_3^h | < n^2 U(2M\delta + t) \qquad (h = 1, 2).$$

If now for all combinations of i, j we fix in each interval of the set u_{ij} a point of e_{ij}, and in the expansion of S_3^1 and S_3^2 we replace $l + (i-1)\eta$ and $l + (j-1)\eta$ by the value of the corresponding function at the point so fixed, we arrive at

$$\sum g_h(\xi_j)\beta_j \qquad (h = 1, 2),$$

where β includes all the intervals of the n^2 sets u_{ij}. From (4) and (6) we get

(8) $$| mE - m\beta | < n^2 t + n^2(2M\delta + t) = \lambda.$$

It is also easy to verify that

(9) $$| S_3^h - S_4^h | < \eta(b - a) = \rho.$$

Combining (5), (7), and (9), we get

(10) $$| S_1^h - S_4^h | < U\lambda + \rho \qquad (h = 1, 2).$$

Then from (a), (b), and (10), we have

$$\left| \int_E g_h dx - \sum_j g_h(\xi_j)\beta_j \right| < \epsilon \qquad (h = 1, 2),$$

and at the same time the right hand side of (8) is less than ϵ', provided first t, and then δ has been fixed sufficiently small.

In the case of p bounded functions the procedure would be the same, except that in the place of the n^2 sets e_{ij} there would be n^p sets similarly defined.

Now suppose the functions $g_1, \cdots, g_p$ unbounded but summable on E. For N a positive integer let E_N be the part of E at which $-N \leqq g_h \leqq N$ $(h = 1, 2, \cdots, p)$. It is clear that E_N is measurable, and that the limit as N becomes infinite of E_N is all the points of E. Hence for N sufficiently large, we have

$$(1) \qquad | mE - mE_N | < \epsilon'/2,$$

and at the same time

$$(2) \qquad | \int_E g_h dx - \int_{E_N} g_h dx | < \epsilon/2 \qquad (h = 1, 2, \cdots, p).$$

The functions g_h are bounded over E_N, and we have seen that if the norm of α is sufficiently small, then from α we can pick a sub-set of intervals β and points ξ_j of E_N, for which

$$(3) \qquad | \int_{E_N} g_h dx - \sum_j g_h(\xi_j)\beta_j | < \epsilon'/2 \quad (h = 1, 2, \cdots, p),$$

and at the same time

$$(4) \qquad | ME_N - m\beta | < \epsilon'/2.$$

Combining (2) with (3), and (1) with (4), we get the first part of the theorem.

Now let E be the interval (a, b). It is evident that there exists $\delta > 0$ such that if e is any measurable set on (a, b) with $me < \delta$, then

$$\int_e \{ | g_1 | - \cdots - | g_p | \} dx < \epsilon/3.$$

Also, on account of the first part of the theorem, if the norm of α is sufficiently small, we have β a sub-set of α such that

$$(1) \qquad | \int_a^b g_h dx - \sum_j 2 g_h(\xi_j)\beta_j | < \epsilon/2 \qquad (h = 1, 2, \cdots, p),$$

and at the same time $| b - a - m\beta | < \delta$. If $\gamma = \alpha - \beta$, then $m\gamma < \delta$, and it is easy to show that there exists ξ_k on each interval of γ with length γ_k such that

$$\sum_k \{ | g_1(\xi_k) | + \cdots + | g_p(\xi_k) | \} \gamma_k \leqq \int_\gamma \{ | g_1 | + \cdots + | g_p | \} dx.$$

Hence

$$| \sum_k g_h(\xi_k)\gamma_k | < \epsilon/2 \qquad (h = 1, 2, \cdots, p),$$

and this with (1) gives

$$| \int_a^b g_h dx - \sum_i g_h(\xi_i)\alpha_i / < \epsilon \quad (h = 1, 2, \cdots, p).$$

None of the foregoing results hold, in general, for an infinite sequence of functions. We give three examples which throw light on this point from various angles.

Let $g_n = n(1 - nx)$ on $0 < x \leqq 1/n$, and $g_n = 0$ elsewhere on $(0, 1)$. Then for every n we have

$$\int_0^1 g_n dx = 1/2.$$

It is easily seen, however, that for any sub-division α whatever of $(0, 1)$, and for any choice of ξ_i, we have

$$\sum_i g_n(\xi_i) = 0$$

for all n sufficiently large.

In this example the sequence converges for each x, but it is not bounded in x and n, nor is it integrable.*

Let $g_{nk} = \log n$ on $k - 1/n \leqq x \leqq k/n$, and $g_{nk} = 0$ elsewhere on $(0, 1)$ $(k = 1, 2, \cdots, n, n = 1, 2, \cdots)$. In this case

$$\int_0^1 g_{nk} dx = \log n/n \leqq 1/e.$$

But if M is arbitrarily large, then for any sub-division whatever of $(0, 1)$ and any choice of ξ_i, it is possible to find g_{nk} so that

$$\sum_i g_{nk}(\xi_i)\alpha_i > M.$$

In this example the sequence is neither bounded in x and n, nor is it convergent. We conclude with an example of a sequence which is bounded in x and n.

Divide the interval $(0, 1)$ into n parts and bisect each of these parts. Let g_{nk} be zero at the irrational points of one-half of each sub-division, and unity at the irrational points of the other half. Doing this in all possible ways gives rise to 2^n functions g_{nk} $(k = 1, 2, \cdots, 2^n)$ on the irrational points of $(0, 1)$. For x rational on $(0, 1)$ let $g_{nk} = 0$ for all n and $k = 1, 2, \cdots, 2^n$. We thus arrive at an infinite sequence of functions bounded in x and n, and such that

$$\int_0^1 g_{nk} dx = \tfrac{1}{2} \qquad (k = 1, 2, \cdots, 2^n, \ n = 1, 2, \cdots).$$

But it is not difficult to show that if $(\xi_1, \xi_2, \cdots, \xi_l)$ are any l irrational numbers on $(0, 1)$ then there exists at least one function g_{nk} such that $g_{nk}(\xi_i) = 0$ $(i = 1, 2, \cdots, l)$. From this it would readily follow that for any sub-division α of $(0, 1)$ and any choice of ξ_i there exists at least one function g_{nk} for which

$$\sum g_n(\xi_i)\alpha_i = 0.$$

We now prove

* Hobson, *Functions of a Real Variable*, second ed., Vol. 2, § 201.

THEOREM II. *Let $g_1, g_2, \cdots$ be a sequence of functions, measurable on (a, b), bounded in x and n, and such that as n increases $g_n(x)$ converges to $g(x)$. Then there exists $\delta > 0$ such that if $\alpha_i < \delta$, it is possible to choose ξ_i so that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

For an arbitrary $\eta > 0$ let $S(\eta, l)$ be the set of x-points for which $| g_n(x) - g(x) | < \eta$ for $n \geqq l$. It can be shown that this set is measurable, and that as l increases the measure of $S(\eta, l)$ approaches $b - a$. This, and the fact that g_n is bounded in x and n allows us to fix $l = l'$ so that,

$$(1) \qquad \left| \int_a^b g_n dx - \int_{S(\eta, l')} g_n dx \right| < \eta \qquad (n = 1, 2, \cdots),$$

and at the same time

$$(2) \qquad b - a - mS(\eta, l') < \eta.$$

Also, we can find $n' \geqq l'$ such that,

$$(3) \qquad \left| \int_{S(\eta, l')} g_n dx - \int_{S(\eta, l')} g_{n'} dx \right| < \eta \qquad (n \geqq n').$$

Then, on account of Theorem I, from α with norm sufficiently small we can choose a sub-set of intervals β, and on each interval of this set a point ξ_j of $S(\eta, l)$ such that both the following inequalities hold:

$$(4) \qquad \left| \int_{S(\eta, l)} g_n dx - \sum_j g_n(\xi_j)\beta_j \right| < \eta \quad (n = 1, 2, \cdots, n').$$

$$(5) \qquad | mS(\eta, l) - m\beta | < \eta.$$

Since $n' \geqq l'$ and ξ_j belongs to $S(\eta, l)$ we have,

$$(6) \qquad \left| \sum_j g_{n'}(\xi_j)\beta_j - \sum_j g_n(\xi_j)\beta_j \right| < \eta(b - a) \qquad (n \geqq n').$$

Since (4) holds for $n = n'$, then by taking into consideration (3) and (6), we have

$$(7) \qquad \left| \int_{S(\eta, l')} g_n dx + \sum_j g_n(\xi_j)\beta_j \right| < 2\eta + \eta(b - a) \qquad (n = 1, 2, \cdots),$$

and this with (1) gives

$$(8) \qquad \left| \int_{S(\eta, l')} g_n dx - \sum_j g_n(\xi_j)\beta_j \right| < 3\eta + \eta(b - a) \qquad (n = 1, 2, \cdots).$$

Let $\gamma = \alpha - \beta$, and let M be the least upper bound of $| g_n |$ for all n and x. Then, on account of (2) and (5), for any choice of ξ_i we have

$$| \sum_k g_n(\xi_k)\gamma_k | < 2\eta M \qquad (n = 1, 2, \cdots),$$

and since η is arbitrary, this with (8) gives the desired result.

If the sequence $g_1, g_2, \cdots$ is not bounded in x and n, but is such that the sequence of integrals is equi-convergent,* then g is summable,† and the sequence of functions is completely integrable.‡ With these facts established, it is not difficult to obtain inequalities (1) and (2) above. We can then proceed as above to inequality (8), thus getting

COROLLARY I. *Let $g_1, g_2, \cdots$ be a sequence of functions, summable on (a, b), convergent to a summable function, and such that the sequence of integrals is equi-convergent. Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to choose from α with norm sufficiently small, a sub-set of intervals β and on each interval of β a point ξ_j such that*

$$| \int_a^b g_n dx - \sum_j g_n(\xi_j)\beta_j | < \epsilon \qquad (n = 1, 2, \cdots),$$

and at the same time $| b - a - m\beta | < \epsilon'$.

The question now arises as to whether or not the conditions of Corollary I are sufficient to permit the approximating sums to be taken over all the intervals of any set α with norm sufficiently small. An example answers this question in the negative. This example also brings to light a class of functions, satisfying the conditions of Corollary I, for which there exists no $\delta > 0$ such that for every α with $\alpha_i < \delta$ it is possible to find ξ_i for which

$$(1) \qquad | \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i | < \epsilon,$$

but for every $\delta > 0$ it is possible to find at least one α with $\alpha_i < \delta$, and a choice of ξ_i so that (1) holds independent of n.

Divide the interval $[1/(k+1), 1/k]$ $(k = 2, 3, \cdots)$ at the points t_{1k}, t_{2k}, $\cdots$ where $t_{1k} = 1/k$, and where $t_{(n-1)k} - t_{nk} = 1/k^5$ so long as the point t_{nk} falls to the right of or on $1/(k+1) + 1/k^5$. If by making $t_{(n-1)k} - t_{nk} = 1/k^5$ the point t_{nk} would fall to the left of $1/(k+1) + 1/k^5$ then place the

* Hobson, **Functions of a Real Variable**, second ed., Vol. 2, § 208.

† de la Vallee Poussin, **Transactions of the American Mathematical Society**, Vol. 15, Theorem I.

‡ Hobson, *loc. cit.*, § 209.

point t_{nk} half way between $t_{(n-1)k}$ and $1/(k+1)$. Let $g_n = k^3$ on $t_{nk}, t_{(n-1)k})$ $(k = 2, 3, \cdots)$, and $g_n = 0$ elsewhere on $(0, 1)$.

It is easily shown that g_n converges to a summable function g, that for all n

$$\int_0^1 g_n dx < \sum_k (1/k^2),$$

and that the sequence of integrals is equi-convergent. But if M is arbitrarily large and δ is any positive number, it is possible to find a sub-division α of $(0, 1)$ with norm less than δ, and a value of n such that

$$\sum g_n(\xi_i)\alpha_i > M. \tag{1}$$

Fix $k = k_1$ so that

$$k^3/k_1(k_1+1) > M, \text{ and } 1/k_1(k_1+1) < \delta.$$

Take any sub-division α of $(0, 1)$ with norm less than δ and such that one of its intervals is $(1/k_1+1, 1/k_1)$. Denote this interval of the set α by α_j, and fix ξ_j any point on α_j. Then for some n we have

$$g_n(\xi_j)\alpha_j \geqq k_1{}^3\alpha_j \geqq k_1{}^3/k_1(k_1+1) \geqq M,$$

and inequality (1) follows from this.

Nevertheless, for ϵ and δ any two positive numbers, it is possible to find a sub-division of $(0, 1)$ with norm less than δ and points ξ_i such that for this particular sub-division and choice of ξ_i we have

$$\left| \int_0^1 g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon.$$

It is easily verified that

$$\int_0^{1/k'} g_n dx \leqq \sum_{k'}^{\infty} (k^3/k^5) \leqq \sum_{k'}^{\infty} (1/k^2) < \epsilon/2 \tag{1}$$

for all n, provided k' is sufficiently large.

On $(1/k', 1)$ g_n is bounded in x and n, and consequently satisfies the conditions of Theorem II. Hence there exists $\delta' < \delta$ such that for any sub-division α' of $(1/k', 1)$ with norm less than δ' it is possible to find ξ_i for which

$$\left| \int_{1/k'}^1 g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon/2 \qquad (n = 1, 2, \cdots). \tag{2}$$

To any such sub-division of $(1/k', 1)$ adjoin the interval $(0, 1/k')$, thus getting a sub-division α of $(0, 1)$. If on this adjoined interval of α we fix

$\xi_1 = 0$, the desired result readily follows from (1), (2), and the fact that $g_n(0) = 0$ for all n.

We now state

THEOREM III. *Let the sequence of functions $g_1, g_2, \cdots$ be summable on (a, b), and converge uniformly to the summable function g. Then on each interval of any finite sub-division α of (a, b) with norm sufficiently small, it is possible to find ξ_i so that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

This readily follows from the uniform convergence of the sequence, and the fact that Theorem I applies to any finite number of functions of the sequence.

A study of the sequence $g_n = \log n/n$ for $0 \leqq x \leqq 1/n$, and $g_n = 0$ elsewhere on $(0, 1)$ shows that the uniform convergence requirement of Theorem III is not necessary.

THEOREM IV. *Let $g_1, g_2, \cdots$ be a sequence of functions summable on (a, b) which converges to the summable function g. Let $\bar{x}$ be the points of non-uniform convergence of the sequence. If $\bar{x}$ has zero content, if g_n is bounded on $\bar{x}$ and n, and if the sequence of integrals is equi-convergent, then there exists α and ξ_i such that*

$$\left| \int_a^b g_n dx - \sum_i g_n(\xi_i)\alpha_i \right| < \epsilon \qquad (n = 1, 2, \cdots).$$

Under the conditions, of the theorem the set $\bar{x}$ can be put in a finite set of intervals β such that if ξ_i is a point of $\bar{x}$ then

$$(1) \qquad \left| \sum_j g_n(\xi_j)\beta_j \right| < \epsilon/3,$$

and at the same time

$$(2) \qquad \left| \int_\beta g_n dx \right| < \epsilon/3.$$

If (a_j, b_j) is one of the p closed intervals of the set complementary to β, then on (a_j, b_j) the sequence converges uniformly to g. Hence by Theorem III, for any sub-division α^j of (a_j, b_j) with norm sufficiently small, we have

$$(3) \qquad \left| \int_{a_j}^{b_j} g_n dx - \sum_i g_n(\xi_i^j)\alpha_i^j \right| < \epsilon/3p \qquad (j = 1, 2, \cdots, p).$$

We can now combine (1), (2), and (3), to obtain the desired result.

Applications to a Function of Two Variables. Let $f(x,y)$ $(a \leqq x \leqq b,$ $c \leqq y \leqq d)$ be bounded, continuous in y for each x, and measurable in x for each y. For δ and η, two arbitrary positive numbers, let $G_{\delta\eta}$ be the set of x-points for which $| f(x,y') - f(x,y'') | < \eta$ for every pair of values y', y'' for which $| \bar{y}' - \bar{y}'' | < \delta$. The set $G_{\delta\eta}$ is measurable.*

To show this let y be an everywhere dense countable set on (a,b), and let $\bar{G}_{\delta\eta}$ be the set of x-points for which $| f(x,\bar{y}') - f(x,\bar{y}'') | < \eta$ for any two values $\bar{y}'$, $\bar{y}''$ of $\bar{y}$ for which $| \bar{y}' - \bar{y}'' | < \eta$. That $\bar{G}_{\delta\eta}$ is measurable readily follows from the countability of $\bar{y}$ and the fact that for a fixed y, $f(x,y)$ is measurable. The continuity of $f(x,y)$ in y for a fixed x can then be used to show that $G_{\delta\eta}$ and $\bar{G}_{\delta\eta}$ are identical. As δ approaches zero, $G_{\delta\eta}$ tends to include all the points of (a,b); this and the boundedness of $f(x,y)$ give, for δ sufficiently small, the following two inequalities:

$$(1) \qquad \left| \int_a^b f(x,y)\,dx - \int_{G_{\delta\eta}} f(x,y)\,dx \right| < \eta$$

$$(2) \qquad b - a - mG_{\delta\eta} < \eta.$$

The function $F(y) = \int_b^a f(x,y)\,dx$ is continuous.†

Hence there exists $\delta' < \delta$ such that

$$(3) \qquad | F(y') - F(y'') | < \eta$$

for any two values y', y'' of y which satisfy $| y' - y'' | < \delta'$. Let (c,d) be divided at the points $y_0 = c$, y_1, $y_2, \cdots, y_p = d$ where $0 < y_k - y_{k-1} < \delta$ $(k = 1, 2, \cdots, p)$. Applying Theorem I to the function $f(x, y_k)$ $(k = 0, 1, \cdots, p)$ and x on $G_{\delta\eta}$, we can select from a finite sub-division of (a,b) with norm sufficiently small, a sub-set β, and ξ_j a point of $G_{\delta\eta}$ such that

$$(4) \qquad \left| \int_{G_{\delta\eta}} f(x,y_k)\,dx - \sum_j f(\xi_j, y_k)\beta_j \right| < \eta \qquad (k = 1, 2, \cdots, p),$$

and at the same time $| m\beta - mG_{\delta\eta} | < \eta$. The above inequalities, together with the boundedness of $f(x,y)$ and the fact that ξ_j belongs to $G_{\delta\eta}$ readily gives the following theorem:

Let $f(x,y)$ satisfy the conditions stated above. Then for any finite sub-

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* Concerning the non-measurability of sets defined in a manner very similar to that of $G_{\delta\eta}$, see Hobson, *loc. cit.*, third ed., Vol. 1, p. 727.

† W. H. Young, *Monatshefte für Mathematik und Physik*, Vol. 21 (1910), pp. 126-127.

division α of (a, b) with norm sufficiently small, we have, for a proper choice of ξ_i,

$$\left| \int_a^b f(x, y)\,dx - \sum_i f(\xi_i, y)\alpha_i \right| < \epsilon.$$

If $f(x, y)$ is unbounded but summable in x for each y, and such that for e any measurable part of (a, b) with me sufficiently small, $\left| \int_e f(x, y)\,dx \right|$ is arbitrarily small independent of y, we shall say that $F(y)$ is equi-convergent. The continuity of $F(y)$ would then follow from Vitali's theorem,* and this gives inequality (2) and (3) above, while (1) would follow from the equi-convergence of $F(y)$. We could then proceed to (4), thus obtaining the following theorem:

Let $f(x, y)$ $(a \leqq x \leqq b,\ c \leqq y \leqq d)$ be continuous in y for each x, summable in x for each y, and such that $F(y)$ is equi-convergent. Then for $\epsilon > 0$ and $\epsilon' > 0$ it is possible to select from α with norm sufficiently small a sub-set β, and on each interval of β a point ξ_j for which

$$\left| \int_a^b f(x, y)\,dx - \sum_j f(\xi_j, y)\beta_j \right| < \epsilon,$$

and at the same time $b - a - m\beta < \epsilon'$.

If it is known that $f(x, y)$ is continuous in y at y_0 only, the other conditions in either the first or second case above remaining unchanged, it does not follow that the set of x-points for which $| f(x, y_0) - f(x, y) | < \eta$ when $| y_0 - y | < \delta$ is measurable.† But if we assume further that $f(x, y)$ is such that this set is measurable for each pair δ and η, it is then possible to establish the first theorem above for some interval about y_0 if $f(x, y)$ is bounded, and the second theorem in case $f(x, y)$ is not bounded but is such that $F(y)$ is equi-convergent.

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. 23 (1907), p. 137.

† Hobson, *loc. cit.*, third ed., Vol. 1, p. 727.

NON-INVOLUTORIAL BIRATIONAL TRANSFORMATIONS BELONGING TO A SPECIAL LINEAR LINE COMPLEX.

By H. A. Davis.

Introduction. The most general non-involutorial Cremona space transformation which belongs to a special linear line complex has been studied synthetically by M. Pieri.* He finds the general transformation to be of order $n + 2n' - 3$. In the present paper a transformation of the same order is discussed which has properties quite different from the one studied by Pieri.

1. *Synthetic Discussion.* Denote by T and Γ respectively the non-involutorial transformation and the special linear complex to which it belongs. Consider two superimposed ordinary spaces Σ and Σ' such that $\Sigma \sim \Sigma'$ under T. To each point P of Σ (or P' of Σ') corresponds the unique Γ-ray PP'. These two representations of the lines of Γ upon the points of Σ and of Σ' shall be designated by M and M' respectively. It is clear that $T = M^{-1}M'$, and $T^{-1} = M'^{-1}M$.†

A Γ-pencil (A, α) with vertex A on the directrix d of Γ and plane α not containing d corresponds in M to a curve $\Delta_{n-1} : A^{n-2}$, and in M' to $\Delta'_{n'-1} : A^{n'-1}$, hence, under T, $\Delta_{n-1} : A^{n-2} \sim \Delta'_{n'-1} : A^{n'-2}$. A Γ-pencil (B, β) with vertex B not on d and plane $\beta : d$ corresponds in M to a conic $\Delta_2 : B$, and in M' to $\Delta_2' : B$, hence, under T, $\Delta_2 : B \sim \Delta_2' : B$. Two pencils, one of each type, with a line in common form a composite Γ-regulus, hence a Γ-regulus R corresponds in M to a curve Δ_{n+1} of genus 0, and in M' to a $\Delta'_{n'+1}$ of genus 0, and, under T, $\Delta_{n+1} \sim \Delta'_{n'+1}$.

The surface F, image in M of a linear Γ-congruence Q_1 with directrix g is cut by a plane through d in d^{n-2} and a Δ_2; and by a plane through g in g and a Δ_{n-1}. Hence, Q_1 corresponds in M to a surface $F_n : d^{n-2}g$, and in M' to $F'_{n'} : d^{n'-2}g$, hence, under T, $F_n : d^{n-2}g \sim F'_{n'} : d^{n'-2}g$.

2. *The Equations of T.* There are in all $\infty^4 \mid F_n \mid$ and $\infty^4 \mid F'_{n'} \mid$, associated with the ∞^4 linear Γ-congruences. But a single pencil $\mid F_n \mid$, together with the corresponding pencil $\mid F'_{n'} \mid$, is sufficient to determine T.

The base of a pencil $\mid Q_1 \mid$ of linear Γ-congruences is a Γ-regulus R on

* "Sulle trasformazioni cirazionali dello spazio inerenti a un complesso lineare speciale, *Circolo Matematico di Palermo*, Vol. 6 (1892), pp. 234-244.

† *loc. cit.*

a quadric H. The directrices of the congruences $| Q_1 |$ form the regulus R' associated with R on H. The directrix d of Γ belongs to R'. Associated with $| Q_1 |$ are the pencils $| F_n |$ and $| F'_{n'} |$.

Through a generic point P of space passes a single surface $F_n : d^{n-2}g$, where g is the directrix of the Q_1 associated with F_n. The corresponding surface is $F'_{n'} : d^{n'-2}g$. The unique transversal of d and g through P meets $F'_{n'}$ in one residual point P', image of P in T.

Denote by $x_1 = x_2 = 0$, $x_1x_3 - x_2x_4 = 0$, $x_1/x_2 + x_4/x_3 = k$, and $x_1/x_4 = x_2/x_3 = m$, the directrix d, the quadric H, the regulus R, and the regulus R' respectively.

If we select the pencils $| F_n |$ and $| F'_{n'} |$ so that neither contains d as a tact locus, we obtain the T discussed by Pieri. We shall consider the case in which d is a tact locus for both pencils. They may then be written

$$(1) \qquad | F_n | \equiv U(x_1 - \rho x_4) + V(x_2 - \rho x_3) = 0,$$

$$(2) \qquad | F'_{n'} | \equiv U'(x_1 - \rho x_4) + V'(x_2 - \rho x_3) = 0,$$

where

$$U = \sum_{i=2}^{n} u_i x_1^{n-i} x_2^{i-2}, \quad V = \sum_{i=2}^{n} v_i x_1^{n-i} x_2^{i-2}, \quad U' = \sum_{i=2}^{n'} u_i' x_1^{n'-i} x_2^{i-2},$$

$$V' = \sum_{i=2}^{n'} v_i' x_1^{n'-i} x_2^{i-2}, \quad u_i = \sum_{k=1}^{4} a_{ik} x_k, \quad v_i = \sum_{k=1}^{4} b_{ik} x_k,$$

$$u_i' = \sum_{k=1}^{4} a'_{ik} x_k, \quad v_i' = \sum_{k=1}^{4} b'_{ik} x_k.$$

Each surface of the pencils $| F_n |$ and $| F'_{n'} |$ contains a line $g \equiv x_1 - \rho x_4 = 0$, $x_2 - \rho x_3 = 0$, of R'. Through a point $P(y)$ of space passes a single surface of $| F_n |$ for which

$$(3) \qquad \rho = [y_1 U(y) + y_2 V(y)]/[y_4 U(y) + y_3 V(y)].$$

The transversal t of d and g through P meets d and g in points whose coördinates are $(0, 0, U, -V)$ and $(\rho y_1, \rho y_2, y_2, y_1)$ respectively. Any point of t has coördinates

$$(4) \qquad x_1 = \mu\rho y_1, \quad x_2 = \mu\rho y_2, \quad x_3 = \mu y_2 + \lambda U, \quad x_4 = \mu y_1 - \lambda V.$$

When (4) is substituted in (2) factors $\lambda\rho$, $(\mu\rho)^{n'-2}$, and $H = y_1y_3 - y_2y_4$ cancel out leaving

$$(5) \qquad \bar{U}'V - \bar{V}'U = 0,$$

where now

$$U = U(y), \quad V = V(y), \quad \bar{U}' = \sum_{i=2}^{n'} \bar{u}_i' y_1^{n'-i} y_2^{i-2}, \quad \bar{V}' = \sum_{i=2}^{n'} \bar{v}_i' y_1^{n'-i} y_2^{i-2},$$

$$\bar{u}_i' = [(a'_{i1}\rho + a'_{i4})y_1 + (a'_{i2}\rho + a'_{i3})y_2]\,\mu + (a'_{i3}U - a'_{i4}V)\lambda,$$

$$\bar{v}_i' = [(b'_{i1}\rho + b'_{i4})y_1 + (b'_{i2}\rho + b'_{i3})y_2]\,\mu + (b'_{i3}U - b'_{i4}V)\lambda,$$

ρ having the value given by (3).

The values of λ and μ obtained from (5) may be written

$$(6) \qquad \lambda = J_1 H + (Uy_1 + Vy_2)K, \quad \mu = J_1(Uy_4 + Vy_3), \qquad \text{where}$$

$$J_1 = U\left(V\frac{\partial U'}{\partial y_3} - U\frac{\partial V'}{\partial y_3}\right) - V\left(V\frac{\partial U'}{\partial y_4} - U\frac{\partial V'}{\partial y_4}\right), \quad K = UV' - U'V.$$

When (6) is substituted in (4) a factor $Uy_1 + Vy_2$ appears, leaving for the equations of T^{-1},

$$(7) \qquad x_1 = y_1 J_1, \quad x_2 = y_2 J_1, \quad x_3 = y_3 J_1 + UK, \quad x_4 = y_4 J_1 - VK.$$

The equations of T are

$$(8) \quad y_1 = x_1 J_1', \; y_2 = x_2 J_1', \; y_3 = x_3 J_1' - U(x)K(x), \; y_4 = x_4 J_1' + V(x)K(x),$$

where

$$J_1' = U'(x)[V'(x)\partial U(x)/\partial x_3 - U'(x)\partial V(x)/\partial x_3]$$
$$- V'(x)[V'(x)\partial U(x)/\partial x_4 - U'(x)\partial V(x)/\partial x_4].$$

It is evident that T aind T^{-1} are of orders $n + 2n' - 3$ and $2n + n' - 3$ respectively.

3. *The System* $\infty^4 \mid F_n \mid$. The Plücker equation of Γ is $p_{12} = 0$. The Plücker coördinates of any Γ-line PP' are

$$p_{12} = 0, \; p_{13} = -Uy_1, \; p_{14} = Vy_1, \; p_{23} = -Uy_2, \; p_{42} = -Vy_2, \; p_{34} = Uy_4 + Vy_3.$$

The ∞^4 linear Γ-congruences $\mid Q_1 \mid$ are the intersections of $p_{12} = 0$ with

$$\alpha_{12}p_{12} + \alpha_{13}p_{13} + \alpha_{14}p_{14} + \alpha_{23}p_{23} + \alpha_{42}p_{42} + \alpha_{34}p_{34} = 0.$$

These $\infty^4 \mid Q_1 \mid$ correspond in M to the $\infty^4 \mid F_n \mid$,

$$(9) \qquad (\alpha_{13}y_1 + \alpha_{23}y_2 - \alpha_{34}y_4)U - (\alpha_{14}y_1 - \alpha_{42}y_2 + \alpha_{34}y_3)V = 0.$$

The directrix g of a Q_1 of this system has coördinates

$$(10) \quad p_{12} = \alpha_{34}, \quad p_{13} = \alpha_{42}, \quad p_{14} = \alpha_{23}, \quad p_{23} = \alpha_{14}, \quad p_{42} = \alpha_{13}, \quad p_{34} = \alpha_{12},$$

$$\alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23} = 0.$$

The equations of g may be written

$$\alpha_{13}y_1+\alpha_{23}y_2-\alpha_{34}y_4=0,\quad \alpha_{14}y_1-\alpha_{42}y_2+\alpha_{34}y_3=0.$$

Each surface F_n of (9) evidently contains d^{n-2} and g, the directrix of its associated congruence.

Consider two surfaces of (9),

$$F_1\equiv(\alpha_{13}y_1+\alpha_{23}y_2-\alpha_{34}y_4)U-(\alpha_{14}y_1-\alpha_{42}y_2+\alpha_{34}y_3)V=0,$$
$$F_2\equiv(\beta_{13}y_1+\beta_{23}y_2-\beta_{34}y_4)U-(\beta_{14}y_1-\beta_{42}y_2+\beta_{34}y_3)V=0.$$

The elimination of U and V from F_1 and F_2 gives the quadric

$$\begin{aligned}H\equiv&(\alpha_{13}y_1+\alpha_{23}y_2-\alpha_{34}y_4)(\beta_{14}y_1-\beta_{42}y_2+\beta_{34}y_3)\\&-(\alpha_{14}y_1-\alpha_{42}y_2+\alpha_{34}y_3)(\beta_{13}y_1+\beta_{23}y_2-\beta_{34}y_4)=0.\end{aligned}$$

This quadric H contains d, g_1 and g_2, the g_1 and g_2 being the directrices of the linear Γ-congruences Q_1 and Q_2, images in M^{-1} of F_1 and F_2. These lines d, g_1 and g_2 determine a regulus R' on H. The Γ-regulus R associated with R' on H is the base of the pencil of linear Γ-congruence defined by Q_1 and Q_2. $[F_1H]=d^{n-2}g\Delta_{n+1}$, where Δ_{n+1} is the image in M of R.

A generic plane $\delta\equiv y_4=\sum_{i=1}^{3}k_iy_i$ meets F_n in a curve $C_n: D^{n-2}$, where $[\delta,d]=D$. In δ, the equation of C_n is

$$(\alpha_{13}y_1+\alpha_{23}y_2-\alpha_{34}\sum_1^3 k_iy_i)U-(\alpha_{14}y_1-\alpha_{42}y_2+\alpha_{34}y_3)V=0,$$

where y_4 in U and V is replaced by $\sum_1^3 k_iy_i$. The tangents to C_n at D as given by the coefficient of $y_3{}^2$ in C_n are

$$k_3\left(\frac{\partial U}{\partial y_3}+k_3\frac{\partial U}{\partial y_4}\right)+\left(\frac{\partial V}{\partial y_3}+k_3\frac{\partial V}{\partial y_4}\right)=0.$$

Since this expression is independent of α_{ik}, it follows that the $\infty^4\,|F_n|$ are mutually tangent in d. Hence, $[F_nF_n]=d^{(n-2)^2+(n-2)}\Delta_{n+1}C_{2n-3}$, where C_{2n-3} and d form the base of the system $\infty^4\,|F_n|$.

4. *The Pencil* $|F_{n-1}|$. Suppose the directrix g of a linear Γ-congruence Q_1 meets d. Then from (10), $\alpha_{34}=0$, and $\alpha_{13}\alpha_{42}+\alpha_{14}\alpha_{23}=0$. The associated surface F_n is then composite, being

(11) $$(\alpha_{13}y_1+\alpha_{23}y_2)(\alpha_{23}U+\alpha_{42}V)=0.$$

The $\alpha_{13}y_1+\alpha_{23}y_2=0$ is the plane (d,g). The surface $F_{n-1}\equiv\alpha_{23}U+\alpha_{42}V$

$= 0$ is the image in M of the Γ-bundle on $G(0, 0, \alpha_{42}, \alpha_{23})$, the point of intersection of d and g.

The section of F_{n-1} by a generic plane $\delta \equiv y_4 = \sum_1^3 k_i y_i$ is a curve $C_{n-1}: D^{n-2}$, where $[d, \delta] = D$. In δ, the equation of C_{n-1} is $\alpha_{23}U + \alpha_{42}V = 0$, where y_4 in U and V is replaced by $\sum_1^3 k_i y_i$. The tangents to C_{n-1} at D as given by the coefficient of y_3 in C_{n-1} are

$$\alpha_{23}\left(\frac{\partial U}{\partial y_3} + k_3\frac{\partial U}{\partial y_4}\right) + \alpha_{42}\left(\frac{\partial V}{\partial y_3} + k_3\frac{\partial V}{\partial y_4}\right) = 0.$$

Since this expression depends upon α_{ik}, it follows that the $\infty^1 \mid F_{n-1} \mid$ are not mutually tangent in d. $[R_{n-1}, F_{n-1}] = d^{(n-2)^2}C_{2n-3}$. This pencil of surfaces $\mid F_{n-1} \mid$, together with the associated pencil $\mid F'_{n'-1} \mid$, image in M' of the ∞^1 Γ-bundles on d, furnishes a simple way of setting up the equations of the T. Let

$$\mid F_{n-1} \mid \equiv \alpha_{23}U + \alpha_{42}V = 0, \quad \mid F'_{n'-1} \mid \equiv \alpha_{23}U' + \alpha_{42}V' = 0.$$

where U, V, U', and V' have the values given in section 2.

Through a generic point $P(y)$ of space passes a single surface F_{n-1} of the pencil $\mid F_{n-1} \mid$, for which $\alpha_{23}/\alpha_{42} = V(y)/-U(y)$. The line t through P and $G(0, 0, \alpha_{42}, \alpha_{23})$ meets $F'_{n'-1}$, image in M' of the bundle G, in a unique point P', image in T of P. Any point of t has coördinates.

$$x_1 = \lambda y_1, \quad x_2 = \lambda y_2, \quad x_3 = \lambda y_3 + \mu U, \quad x_4 = \lambda y_4 - \mu V.$$

When the ratio λ/μ is determined so that this point lies on $F'_{n'-1}$, the result is (7).

5. *The $T_{3,3}$ in a Plane Through d.* A plane $\gamma \equiv x_1 = \sigma x_2$ through d cuts $\mid F_{n-1} \mid \equiv \alpha_{23}U(x) + \alpha_{42}V(x) = 0$ and $\mid F'_{n'-1} \mid \equiv \alpha_{23}U'(x) + \alpha_{42}V'(x) = 0$ in residual pencils of lines

$$\mid l \mid \equiv \alpha_{23}(ax) + \alpha_{42}(bx) = 0 \quad \text{and} \quad \mid l' \mid \equiv \alpha_{23}(a'x) + \alpha_{42}(b'x) = 0$$

respectively, where $(ax) = a_2x_2 + a_3x_3 + a_4x_4$,

$$a_2 = \sum_{i=2}^{n} \sigma a_{i1} + a_{i2})\sigma^{n-i}, \quad a_3 = \sum_{i=2}^{n} a_{i3}\sigma^{n-i}, \quad a_4 = \sum_{i=2}^{n} a_{i4}\sigma^{n-i}, \quad \text{etc.}$$

The vertices of $\mid l \mid$ and $\mid l' \mid$ are respectively $L(\lambda)$ and $L'(\lambda')$, where $\lambda_1 = \sigma\lambda_2$, $\lambda_2 = \mid a_3b_4 \mid$, $\lambda_3 = -\mid a_2b_4 \mid$, $\lambda_4 = \mid a_2b_3 \mid$, $\lambda_1' = \sigma\lambda_2'$, $\lambda_2' = \mid a_3'b_4' \mid$, $\lambda_3' = -\mid a_2'b_4' \mid$, $\lambda_4' = \mid a_2'b_3' \mid$.

Through a generic point $P(y)$ of γ passes one line of $\mid l \mid$, for which

$\alpha_{23}/\alpha_{42} = (by)/-(ay)$. The corresponding line $l' \equiv (by)(a'x) - (ay)(b'x) = 0$ is met by the line through $P(y)$ and $G[0, (ay), -(by)]$ in a point P', image of P in T. The T_3^{-1} in γ is thus found to be

(12) $$x_2 = y_2 j_1, \quad x_3 = y_3 j_1 + (ay)k, \quad x_4 = y_4 j_1 - (by)k,$$

where $j_1 = (ay)[a_3'(by) - b_3'(ay)] - (by)[a_4'(by) - b_4'(ay)]$,

and $k = (ay)(b'y) - (by)(a'y)$.

The T_3 is

(13) $$y_2 = x_2 j_1', \quad y_3 = x_3 j_1' - (a'x)k(x), \quad y_4 = x_4 j_1' + (b'x)k(x),$$

where $j_1' = (a'x)[a_3(b'x) - b_3(a'x)] - (b'x)[a_4(b'x) - b_4(a'x)]$.

The conic $k : LL'$ is pointwise invariant under $T_{3,3}$.

The points L and L' are evidently fundamental under T_3 and T_3^{-1} respectively.

The points P_1' and P_2', intersection residual to L' of $k(x) = 0$ with the pair of lines $j_1'(x) = 0$, are fundamental for T_3^{-1}.

Since (13) may be written

$$y_2 = x_2 j_1'(x),$$
$$y_3 = [x_4(a'x) + x_3(b'x)][b_4(a'x) - a_4(b'x)] + [b_2(a'x) - a_2(b'x)]x_2(a'x),$$
$$y_4 = -[x_4(a'x) + x_3(b'x)][b_3(a'x) - a_3(b'x)] + [b_2(a'x) - a_2(b'x)]x_2(b'x),$$

it follows that the points P_3', P_4', intersection of the conic $x_4(a'x) + x_3(b'x) = 0$ with the line $x_2 = 0$, are fundamental for T_3^{-1}.

The homaloidal nets of T_3 and T_3^{-1} are respectively

$$\infty^2 \mid f_3' \mid : L'^2 P_1' P_2' P_3' P_4', \quad \text{and} \quad \infty^2 \mid f_3 \mid : L^2 P_1 P_2 P_3 P_4.$$

The images in T_3^{-1} of $(a'x) = 0$, $k(x) = 0$, and $f_3'(x) = 0$ are respectively $(ay)j_2 = 0$, $kj_2j_3 = 0$, and $(\alpha y)j_1 j_2^2 j_3 = 0$. The factors $(ay) = 0$, $k = 0$, $(\alpha y) = 0$, $j_1 = 0$, $j_2 = 0$, and $j_3 = 0$ are the images respectively of the proper points of $(a'x) = 0$, the proper points of $k(x) = 0$, the proper points of $f_3'(x) = 0$, the pair of points $P_3'P_4'$, the point L', and the pair of points $P_1'P_2'$.

$$j_2 = (ay)[a_3'(b'y) - b_3'(a'y)] - (by)[a_4'(b'y) - b_4'(a'y)],$$
$$j_3 = (ay)[a_3(by) - b_3(ay)] - (by)[a_4(by) - b_4(ay)].$$

The images in T_3^{-1} of $j_1'(x) = 0$, $j_2'(x) = 0$, and $j_3'(x) = 0$ are respectively $j_2^2 j_3 = 0$, $j_1 j_2 j_3 = 0$, and $j_1 j_2^2 = 0$. The jacobian of T_3^{-1} is now seen to be made up of $j_1 : L^2 P_1 P_2$, $j_2 : LL'P_1P_2P_3P_4$, and $j_3 : L^2 P_3 P_4 P_1' P_2'$. Similarly the jacobian of T_3 is composed of $j_1' : L'^2 P_1' P_2'$, $j_2' : LL'P_1'P_2'P_3'P_4'$, and $j_3' : L'^2 P_3' P_4' P_1 P_2$.

As the plane γ generates the pencil on d, the $T_{3,3}$ generates the space $T_{n+2n'-3,2n+n'-3}$. The equations of the latter may readily be obtained from (12) and (13) by replacing (ay), (by), $(a'x)$, $(b'x)$, and σ by U, V, $U'(x)$, $V'(x)$, *and* x_1/x_2 respectively.

Since the point L is the section by γ of C_{2n-3}, the latter may be represented by $x_1 = \sigma\lambda_2$, $x_2 = \lambda_2$, $x_3 = \lambda_3$, $x_4 = \lambda_4$. Similarly, $C'_{2n'-3}$ may be written $x_1 = \sigma\lambda_2'$, $x_2 = \lambda_2'$, $x_3 = \lambda_3'$, $x_4 = \lambda_4'$.

6. *The Homaloidal Webs in the* $T_{n+2n'-3,2n+n'-3}$. The image in T^{-1} of a generic plane $\sum_{i=1}^{4} c_i x_i = 0$ is a surface $F_{2n+n'-3} \equiv J_1 \sum_{i=1}^{4} c_i y_i + (c_3 U - c_4 V) K = 0$. A plane $\delta \equiv y_4 = \sum_{i=1}^{3} k_i y_i$ meets $F_{2n+n'-3}$ in a curve $C: D^{2n+n'-6}$, where $[\delta, d] = D$. The tangents to C at D as given by the coefficient of $y_3{}^3$ in C are

$$(k_3\bar{U} + \bar{V}) \left[c_3 \left(\bar{U} \frac{\partial V'}{\partial y_4} - \bar{V} \frac{\partial U'}{\partial y_4} \right) - c_4 \left(\bar{U} \frac{\partial V'}{\partial y_3} - \bar{V} \frac{\partial U'}{\partial y_3} \right) \right] = 0,$$

where

$$\bar{U} = \partial U/\partial y_3 + k_3 \partial U/\partial y_4, \quad V = \partial V/\partial y_3 + k_3 \partial V/\partial y_4.$$

Since the factor $k_3\bar{U} + \bar{V}$ is independent of c_i, it follows that, of the $2n + n' - 6$ sheets of the surfaces of the homaloidal web through d, $n - 2$ are mutually tangent there. It follows that the homaloidal web of T^{-1} is

$$\infty^3 \mid F'_{n+2n'-3} \mid : d^{n+2n'-6} C'^2_{2n'-3} \gamma'_{4n+2n'-10}.$$

Similarly, that of T is

$$\infty^3 \mid F_{2n+n'-3} \mid : d^{2n+n'-6} C^2_{2n-3} \gamma_{2n+4n'-10}.$$

7. *The F- and P-Systems of* $T_{2n+n'-3,n+2n'-3'}$. As the plane γ of section 5 describes the pencil on d, the points L, L', P_1P_2, $P_1'P_2'$, P_3P_4, and $P_3'P_4'$ describe respectively the curves C_{2n-3}, $C'_{2n'-3}$, $\gamma_{2n+4n'-10}$, $\gamma'_{4n+2n'-10}$, d and d. It is evident that C_{2n-3} (or $C'_{2n'-3}$) meets d in $2n - 4$ (or $2n' - 4$) points, and is rational. Also, $\gamma_{2n+4n'-10}$ (or $\gamma'_{4n+2n'-10}$) meets d in $2n + 4n' - 12$ (or $4n + 2n' - 12$) points. It is of genus $n' - 3$ (or $n - 3$).

The images in T^{-1} of d, $U'(x) = 0$, and $K(x) = 0$ are respectively $J_1 = 0$, $UJ_1{}^{n'-2}J_2 = 0$, and $KJ_1{}^{n+n'-4}J_2J_3 = 0$, where

$$J_1 = U\left(V \frac{\partial U'}{\partial y_3} - U \frac{\partial V'}{\partial y_3} \right) - V\left(V \frac{\partial U'}{\partial y_4} - U \frac{\partial V'}{\partial y_4} \right),$$

$$J_2 = U\left(V' \frac{\partial U'}{\partial y_3} - U' \frac{\partial V'}{\partial y_3} \right) - V\left(V' \frac{\partial U'}{\partial y_4} - U' \frac{\partial V'}{\partial y_4} \right),$$

$$J_3 = U\left(V \frac{\partial U}{\partial y_3} - U \frac{\partial V}{\partial y_3} \right) - V\left(V \frac{\partial U}{\partial y_4} - U \frac{\partial V}{\partial y_4} \right).$$

The factors $U=0$, $K=0$, $J_1=0$, $J_2=0$, and $J_3=0$ are the images respectively of the proper points on $U'(x)=0$, the proper points on $K(x)=0$, the line d, the curve $C'_{2n'-3}$, and the curve $\gamma'_{4n+2n'-10}$.

These surfaces $J_i=0$ can also be obtained from the curves $j_i=0$ of section 5.

The jacobian of T^{-1} is composed of

$$\begin{aligned} &J_1^2: d^{2n+n'-6}C^2_{2n-3}\gamma_{2n+4n'-10},\\ &J_2: d^{n+2n'-6}C_{2n-3}C'_{2n'-3}\gamma_{2n+4n'-10},\\ &J_3: d^{3n-6}C^2_{2n-3}\gamma'_{4n+2n'-10}.\end{aligned}$$

The J_1, J_2, and J_3 are of orders $2n+n'-4$, $n+2n'-4$, and $3n-4$ respectively.

The $J_1=0$ and $J_3=0$ are ruled Γ-surfaces generated by the pairs of lines $j_1=0$ and $j_3=0$ of section 5. The $J_2=0$ is not ruled but is generated by the conic $j_2=0$. The image in T^{-1} of a line g is a curve of order $n+2n'-3$. If g meets d, its image in the T_3^{-1} in the plane (g, d) is a cubic curve. Hence, a point of d corresponds in T^{-1} to a curve of order $n+2n'-6$. The $J_1=0$ contains a single infinity of such curves.

The pointwise invariant surface is

$$K_{n+n'-2}: d^{n+n'-4}C_{2n-3}C'_{2n'-3}\gamma_{2n+4n'-10}\gamma'_{4n+2n'-10}.$$

A plane $\delta \equiv y_4 = \sum_{i=1}^{3} k_i y_i$ meets $J_2=0$ in a curve $C_2: D^{n+2n'-6}$, where $[\delta, d]=D$. The tangents to C_2 at D as given by the coefficient of y_3^2 in C_2 are

$$\left(\frac{\partial U}{\partial y_3}\frac{\partial V}{\partial y_4}-\frac{\partial U}{\partial y_4}\frac{\partial V}{\partial y_3}\right)\left[k_3\left(\frac{\partial U}{\partial y_3}+k_3\frac{\partial U}{\partial y_4}\right)+\left(\frac{\partial V}{\partial y_3}+k_3\frac{\partial V}{\partial y_4}\right)\right]=0.$$

Similarly, the tangents to $C_3=[J_3, \delta]$ at D are

$$\left(\frac{\partial U'}{\partial y_3}\frac{\partial V'}{\partial y_4}-\frac{\partial U'}{\partial y_4}\frac{\partial V'}{\partial y_3}\right)\left[k_3\left(\frac{\partial U}{\partial y_3}+k_3\frac{\partial U}{\partial y_4}\right)+\left(\frac{\partial V}{\partial y_3}+k_3\frac{\partial V}{\partial y_4}\right)\right]=0.$$

It follows that $J_2=0$ and $J_3=0$ are tangent $n-2$ times in d.

$$[J_1, J_2]=d^{(2n+n'-6)(n+2n'-6)}C^2_{2n-3}\gamma_{2n+4n'-10}(2n'-4)l_i,$$

where the $(2n'-4)l_i$ are the images in T^{-1} of the $(2n'-4)$ points of intersection of $C'_{2n'-3}$ and d.

$$[J_1, J_3]=d^{(2n+n'-6)(3n-6)}C^4_{2n-3}(2n+2n'-8)l_i,$$

where the l_i are the images in T^{-1} of $(2n+2n'-8)$ of the points of inter-

section of $\gamma'_{4n+2n'-10}$ and d. Each of the remaining $2n-4$ points of intersection corresponds to d itself and is already counted.

$$[J_2, J_3] = d^{(n+2n'-6)(3n-6)+n-2}C^2_{2n-3}(3n+4n'-12)l_i,$$

where the l_i are the images of the points of intersection of $C'_{2n'-3}$ and $\gamma'_{4n+2n'-10}$.

$$[J_1, K] = d^{(2n+n'-6)(n+n'-4)}C^2_{2n-3}\gamma_{2n+4n'-10},$$
$$[J_2, K] = d^{(n+2n'-6)(n+n'-4)}C_{2n-3}C'_{2n'-3}\gamma_{2n+4n'-10},$$
$$[J_3, K] = d^{(3n-6)(n+n'-4)}C^2_{2n-3}\gamma'_{4n+2n'-10}.$$

The jacobian of T is composed of

$$J_1'^2 : d^{n+2n'-6}C'^2_{2n'-3}\gamma'_{4n+2n'-10},$$
$$J_2' : d^{2n+n'-6}C_{2n-3}C'_{2n'-3}\gamma'_{4n+2n'-10},$$
$$J_3' : d^{3n'-6}C'^2_{2n'-3}\gamma_{2n+4n'-10}, \quad \text{where}$$

$$J_1' = U'\left(V'\frac{\partial U}{\partial x_3} - U'\frac{\partial V}{\partial x_3}\right) - V'\left(V'\frac{\partial U}{\partial x_4} - U'\frac{\partial V}{\partial x_4}\right),$$
$$J_2' = U'\left(V\frac{\partial U}{\partial x_3} - U\frac{\partial V}{\partial x_3}\right) - V'\left(V\frac{\partial U}{\partial x_4} - U\frac{\partial V}{\partial x_4}\right),$$
$$J_3' = U'\left(V'\frac{\partial U'}{\partial x_3} - U'\frac{\partial V'}{\partial x_3}\right) - V'\left(V'\frac{\partial U'}{\partial x_4} - U'\frac{\partial V'}{\partial x_4}\right),$$

where $U = U(x)$, etc.

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ON THE POSSIBLE FORMS OF DISCRIMINANTS OF ALGEBRAIC FIELDS I.

By William R. Thompson.

Let $K_{(\theta)}$ be an algebraic field of n-th degree, and let d be the discriminant of the field. Furthermore, let

$$(1)\qquad p = \prod_{i=1}^{r} P_i^{e_i}, \quad \text{where} \quad N_{(P_i)} = p^{f_i},$$

be the prime-ideal decomposition of an arbitrary rational prime, $p > 1$, in the field. Then Dedekind * has shown that the rational integer, $\varepsilon \geq 0$, such that p^ε exactly divides d, is dependent upon (1) and that if none of the exponents (e_i) are divisible by p, then

$$(2)\qquad \varepsilon = \sum_{i=1}^{r} f_i(e_i - 1).$$

Ore † has treated the general case, where all or any of the exponents (e_i) may be divisible by p, and given the possible values of ε for any given prime-ideal decomposition. Furthermore, he has given the maximal value of ε for fields of n-th degree.

Let $N_{(n,p)}$ denote this maximal value. Then if we have the representation of n in a p-adic system

$$(3)\qquad n = \sum_{\alpha=0}^{q} b_\alpha p^\alpha, \quad \text{where} \quad 0 \leq b_\alpha < p$$

and b_α is a rational integer; and J is the aggregate number of these coefficients (b_α) which are different from zero; then Ore has shown that

$$(4)\qquad N_{(n,p)} = \sum_{\alpha=0}^{q} [b_\alpha(\alpha + 1)p^\alpha] - J.$$

The equivalent of the above is given in the paper ‡ previously mentioned wherein Ore has suggested the interest of ascertaining what other values are possible for ε for algebraic fields of the same degree. It is the purpose of

* R. Dedekind, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 29 (1882), pp. 1-56.

† Ö. Ore, *Mathematische Annalen*, Vol. 96 (1927), pp. 313-352.

‡ Ore, *loc. cit.*

the present communication to present the solution of this problem, which may be stated briefly as follows:

A. If $p > 2$, ε can assume all values from zero to $N_{(n,p)}$ inclusive except only as follows:

$\varepsilon \neq \alpha p^a - 1$ (where α is a positive rational integer) if $n = p^a$ or if $\alpha > 1$ and $n = p^a + 1$.

B. If $p = 2$ the result is formally the same as given for $p > 2$ except that $\varepsilon \neq 1$ always.

During the course of the proof of these relations which follow, it will become evident that certain other relations are developed which make possible a partial reversal of proof in order to establish criteria whereby, in certain cases, a knowledge of the power of p exactly dividing the discriminant of a field of n-th degree suffices to determine uniquely the prime-ideal decomposition of p in *any* such field. The statement and proof of these relations, however, will be given in another communication.

1. Let $K_{(\theta)}$, n, d, p, ε and $N_{(n,p)}$ be defined as above; and let p have the prime-ideal decomposition (1). Then let the representation of n in the p-adic system be given as in (3) and accordingly $N_{(n,p)}$ will be given by (4). Then it is well known that

$$n = \sum_{i=1}^{r} e_i f_i, \quad \text{and} \quad e_i > 0 < f_i, \tag{5}$$

As indicated above, the foundation of the proof to follow is found in certain theorems given by Ore.* These may be stated in the following form:

For each prime-ideal, P_i, there exists a rational integer $\rho_i \geq 0$, (called the *supplemental* number) such that if $S_i \geq 0$ be a rational integer such that e_i is exactly divisible by p^{S_i} then ρ_i is determined as follows:

(6) If $S_i = 0$, then $\rho_i = 0$; and if $S_i \neq 0$, then $1 \leq \rho_i \leq e_i S_i$, and in this latter alternative ρ_i is restricted by the condition that if there exists a positive rational integer, ν_i, such that ρ_i is exactly divisible by p^{ν_i}, then ν_i shall not exceed ρ_i/e_i, i. e., $\nu_i \leq [\rho_i/e_i]$, and in any case,

$$\varepsilon = \sum_{i=1}^{r} f_i(e_i - 1 + \rho_i). \tag{7}$$

This theorem will be designated as Ore's First Theorem. The following existance theorem given in the same article will be called Ore's Second Theorem:

* *Loc. cit.*

For any set of rational integers, designated by

$$\left\{\begin{matrix} e_1, \cdots, e_r \\ f_1, \cdots, f_r \\ \rho_1, \cdots, \rho_r \end{matrix}\right\} \tag{8}$$

and satisfying the conditions of (5) *and* (6) *there exists an algebraic field of n-th degree such that its discriminant is exactly divisible by p^ε where ε has the value given in* (7). *Furthermore, in such a field the prime-ideal decomposition of p is given by* (1).

2. We may proceed from these two theorems and that relative to $N_{(n,p)}$ given in (3) and (4) to find what other values of ε are attainable for fields of n-th degree. Accordingly, let $\mathcal{E}_p^{(n)}$ be defined as the set of rational integers which are attainable values of ε for algebraic fields of n-th degree. Then, obviously,

(9) $N_{(n,p)}$ is the greatest component of $\mathcal{E}_p^{(n)}$.

Now in (8), obviously, by taking $e_i = 1$ for every i, then $S_i = 0$ whence by (6) $\rho_i = 0$ for every i whence (7) gives $\varepsilon = 0$, whence

(10) 0 is a component of $\mathcal{E}_p^{(n)}$ for every n and p.

(9) and (10) by Ore's Theorems then give the obvious but useful

THEOREM 1. *$\mathcal{E}_p^{(n)}$, the set of attainable values of ε for fields of n-th degree, is a set of a finite number of rational integers including* 0 *and $N_{(n,p)}$ as least and greatest component respectively.*

Obviously, there are instances [where $n = 1$ in (3)] where these two extremes are equal, in which case 0 is the only component of $\mathcal{E}_p^{(n)}$ but the set is never *void.*

Now consider the case, $p > n$. Then by (4) we have

$$N_{(n,p)} = n - 1; \tag{11}$$

and in (8) we may choose $f_i = 1$ for every i; and as $p > n$, (5) gives $p > e_i$ whence by Ore's First Theorem or that of Dedekind we have in this case

$$\varepsilon = n - r \tag{12}$$

where r can be any positive rational integer not exceeding n whence by Theorem 1 and (11) we have proved the

THEOREM 2. *For $p > n$, $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$.*

3. Now, before we attempt the proof of the general theorem stated in the introduction, it is expedient to prove certain incidental theorems some of

which have a wider application than that to be utilized at present, as has been mentioned above.

Consider the set of numbers in (8). It consists of three arrays of rational integers, each containing r numbers; the three arrays (the orders, degrees and supplemental numbers, respectively, of the prime-ideal divisors of p) being arranged in the *form* of a matrix. Obviously any permutation of the columns of this matrix gives the same value to ε. Such a matrix satisfying the conditions given in (5) and (6) as indicated under (8) will be called a *critical matrix*. Furthermore, it is evident that the set of possible values for each ρ_i depends upon p and e_i only.

Accordingly, if $r > 1$, we may take any positive rational integer, $r' < r$, and form the two critical matrices for fields of degree less than n

$$(13) \qquad \left\{ \begin{matrix} e_1, \cdots, e_{r'} \\ f_1, \cdots, r_{r'} \\ \rho_1, \cdots, \rho_{r'} \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} e_{r'+1}, \cdots, e_r \\ f_{r'+1}, \cdots, f_r \\ \rho_{r'+1}, \cdots, \rho_r \end{matrix} \right\}.$$

Now, let n', n'', ε' and ε'' be defined by

$$(14) \qquad n' = \sum_{i=1}^{r'} e_i f_i, \quad n'' = \sum_{i=r'+1}^{r} e_i f_i,$$

$$\varepsilon' = \sum_{i=1}^{r'} f_i(e_i - 1 + \rho_i), \quad \text{and} \quad \varepsilon'' = \sum_{i=r'+1}^{r} f_i(e_i - 1 + \rho_i).$$

Then by (5) and (7) we have

$$(15) \qquad n = n' + n'' \quad \text{and} \quad \varepsilon = \varepsilon' + \varepsilon''.$$

Now, in the present discourse, if A and B are sets of rational integers, let $A + B$ be assigned the following meaning:

(16) If η is a component of $A + B$; then there exists a component of A, say a, and a component of B, say b, such that $\eta = a + b$;
and if a is a component of A, and b is a component of B, then $(a + b)$ is a component of $(A + B)$.

Now, if we have any two critical matrices whatever, we may write them in the form of (13) and employ the definitions under (14); but with no prior assumption as to the matrix in (8). However, let n' and n'' be restricted by the equation

$$n' + n'' = n$$

then, obviously, by a reversal of proof we may construct the critical matrix (8) by the fusion of those of (13) and thus establish the existence of a field of n-th degree wherein the discriminant is exactly divisible by $p^{\varepsilon'+\varepsilon''}$, i. e.,

$$\varepsilon = \varepsilon' + \varepsilon''.$$

Accordingly, we have proved the

THEOREM 3. *If n' and n'' are two positive rational integers such that their sum is equal to n then $\mathcal{E}_p^{(n)}$ includes $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')}$.*

Also, inasmuch as by Theorem 1 zero is always a component of $\mathcal{E}_p^{(n)}$ for any n, we have to Theorem 3 the

COROLLARY 1. $\mathcal{E}_p^{(n)}$ includes $\mathcal{E}_p^{(n')}$ provided $n \geqq n'$.

4. In order to facilitate calculation let us divide the set, $\mathcal{E}_p^{(n)}$ dichotomously as follows:

Let $H_p^{(n)}$ be a set such that if and only if η is a component of $H_p^{(n)}$ then there exist two positive rational integers, n' and n'', such that $n = n' + n''$ and η is a component of $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')}$.

Obviously, by Theorem 3, therefore

(17) $\mathcal{E}_p^{(n)}$ includes $H_p^{(n)}$.

Accordingly, let $A_p^{(n)}$ be defined as the set of rational integers such that if and only if η is a component of $\mathcal{E}_p^{(n)}$ but not of $H_p^{(n)}$, then η is a component of $A_p^{(n)}$.

Then $\mathcal{E}_p^{(n)}$ is the union of the two mutually exclusive sets, $H_p^{(n)}$ and $A_p^{(n)}$, which we may call the *heritage* and the *acquisition* respectively of $\mathcal{E}_p^{(n)}$.

Now, by (13), (14) and (15) and the above definitions any value of ε obtained from a critical matrix (8) wherein $r > 1$ lies in the heritage, $H_p^{(n)}$, and not in $A_p^{(n)}$. Otherwise stated we have proved

(18) if ε is a component of $A_p^{(n)}$, then $r = 1$.

Now in (8) let us consider the case $r = 1$ and let the superfluous subscript, i, then be dropped. Then the critical matrix becomes simply

$$(19) \qquad \left\{ \begin{matrix} e \\ f \\ \rho \end{matrix} \right\} \quad \text{and} \quad \varepsilon = f(e - 1 + \rho).$$

Now, if in (19) $f > 1$ then there exists another critical matrix under (8) where $r = 2$, namely

$$(20) \qquad \left\{ \begin{matrix} e, & e \\ 1, & f - 1 \\ \rho, & \rho \end{matrix} \right\} \quad \text{and} \quad \varepsilon = f(e - 1 + \rho),$$

which value of ε is the same as in (19) but by (18) this is not a component of $A_p^{(n)}$. Accordingly, by (18) we have in (8)

(21) if ε is a component of $A_p^{(n)}$, then $r = f_i = 1$, and $e_i = n$..

Therefore, by Ore's First Theorem we have the

THEOREM 4. *If ε is a component of $A_p^{(n)}$ it corresponds to a critical matrix of the form*

$$\left\{ \begin{matrix} n \\ 1 \\ \rho \end{matrix} \right\} \quad \textit{and} \quad \varepsilon = n - 1 + \rho,$$

where ρ is a rational integer defined by the relations:

If $S \geqq 0$ is a rational integer such that n is exactly divisible by p^S, then

$$\begin{aligned} &\textit{if } S = 0, \quad \textit{then} \quad \rho = 0, \\ &\textit{and if } S \neq 0, \quad \textit{then} \quad 1 \leq \rho \leq nS \end{aligned}$$

and in this latter alternative ρ is restricted by the condition that if there exists a positive rational integer, ν, such that ρ is exactly divisible by p^ν, then ν shall not exceed ρ/n.

Accordingly, by the definition of $A_p^{(n)}$ and Ore's Theorems we have

THEOREM 5. *That ε be a component of $A_p^{(n)}$ it is necessary and sufficient that the conditions of Theorem 4 be satisfied and that ε be not a component of $H_p^{(n)}$.*

5. In Section 2 we have obtained a solution for the case $p > n$ and in the succeeding sections have prepared for certain phases of the handling of the other cases $(p \leq n)$. However, before attempting the general solution there are a few additional contingencies for which provision should be made.

In order to illustrate this need as well as to extend the domain of the solution, let us consider the case, $p = n$. Then by Theorem 2 and the definition of $H_p^{(n)}$ we have

(22) $$H_p^{(p)} = \mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} \quad \text{where} \quad n' + n'' = p$$

whence, by the definition in (16), we have

(23) $$H_p^{(p)} = 0, \cdots, p - 2.$$

Now in Theorem 4 for $n = p$ we have

(24) $$S = 1 \quad \text{and} \quad \rho = 1, \cdots, p \quad \text{and}$$

(25) $$A_p^{(p)} = p, \cdots, 2p - 1$$

whence (as $N_{(p,p)} = 2p - 1$) we have by the definitions of Section 4 the

THEOREM 6. $\mathcal{E}_p^{(p)} = 0, \cdots, N_{(p,p)}$ *except* $p - 1$.

Here we note the first instance of a number, η, a rational integer such that

(26) $0 \leq \eta \leq N_{(n,p)}$ and yet η is not a component of $\mathcal{E}_p^{(n)}$.

Such a number will be called an *exceptional number* relative to $\mathcal{E}_p^{(n)}$. Furthermore, in (26) by Theorem 1 we have $0 \neq \eta \neq N_{(n,p)}$.

Now, if η in (26) is also an exceptional number relative to $\mathcal{E}_p^{(\mu)}$ for every $\mu > n$ (where μ is a positive rational integer) then we shall call η a *universal exception* relative to p. Obviously, by Cor. 1 of Theorem 3, then η is not a component of $\mathcal{E}_p^{(\mu)}$ for any positive rational integer, μ.

On the other hand, if η is an exceptional number relative to $\mathcal{E}_p^{(n)}$ but $\eta - 2, \eta - 1, \eta + 1$ and $\eta + 2$ are components of $\mathcal{E}_p^{(n)}$; then η will be called a *regular exception* relative to $\mathcal{E}_p^{(n)}$.

Obviously, by Theorem 6 we have

(27) $\mathcal{E}_p^{(p)}$ has the single exceptional number, $p - 1$,

which is regular for $p > 2$; and for $p = 2$ the exception ($\eta = 1$) is not regular.

Now, suppose that 1 is not a component of $\mathcal{E}_2^{(\mu)}$ for $\mu < n$. Then by definition 1 is not in $H_2^{(n)}$; and by Theorem 4

1 is not in $A_2^{(n)}$; whence

1 is not in $\mathcal{E}_2^{(n)}$; whence, obviously, by complete induction we have the

THEOREM 7. *The number 1 is a universal exception relative to the rational prime, 2; that is, the number 1 is not a component of $\mathcal{E}_2^{(n)}$ for any n.*

Obviously, from the definitions we have to this theorem the

COROLLARY 1. $d \not\equiv 2 \pmod 4$.

This is merely a restatement of Theorem 7 and is essentially the same as part of a result obtained by Stickelsberger * concerning the *discriminant of the equation, D.* The equation here as usual being by implication the equation of θ, there exists the well known relation

(28) $$D = k^2 d$$

* L. Stickelsberger, *Proceedings of the International Congress,* Zürich (1897), pp. 182-193.

where k is a rational integer called the index of θ. Obviously, then by the corollary above we have

$$(29) \qquad D \not\equiv 2 \pmod 4)$$

which is a part of the result of Stickelsberger mentioned above. Another proof of the Theorem of Stickelsberger has recently been given by I. Schur.*

Now, let us assume that for any two positive rational integers, n' and n'', such that $n' + n'' = n$, the sets $\mathcal{E}_p^{(n')}$ and $\mathcal{E}_p^{(n'')}$ have at most one regular exception each and no other exceptions unless $p = 2$ and that in this case the only other exception is the universal exception, 1. Then, obviously, by Theorems 2 and 6 we have

$$(30) \quad \text{If } n' > 1 < n'', \text{ then } \mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} = 0, \cdots, (N_{(n',p)} + N_{(n'',p)}) \text{ except 1 if } p = 2.$$

and if either n' or $n'' = 1$, then $\mathcal{E}_p^{(n')} + \mathcal{E}_p^{(n'')} = \mathcal{E}_p^{(n-1)}$.

6. We are now prepared to prove by the method of complete induction the following general theorem.

THEOREM 8. *If α is a positive rational integer and $p > 2$; then, if $n = p^\alpha$, then $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$ except $\alpha p^\alpha - 1$, if $\alpha > 1$* and $n = p^\alpha + 1$, *then $\mathcal{E}_p^{(n)} = \mathcal{E}_p^{(n-1)}$, and in every other case $\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)}$; and if $p = 2$ then $\mathcal{E}_p^{(n)}$ is formally the same as given for $p > 2$ except that* 1 *is a universal exception.*

Now, by Theorem 2 and 6 we have verified Theorem 8 for the case, $n \leq p$.

Let k be a positive rational integer such that Theorem 8 is verified for the case, $n \leqq p^k$. By the statement preceding there is at least one possible value for k; namely, the number 1. It remains, accordingly, but to establish that given a value for k above then the Theorem 8 can be verified for the case, $n \leq p^{k+1}$; and, obviously, it suffices to make the demonstration for the case, $p^k + 1 \leq n \leq p^{k+1}$; and in so doing we may refer to Theorem 8 for the enunciation of the components of any set, $\mathcal{E}_p^{(\mu)}$, provided that μ is a positive rational integer not exceeding p^k. Furthermore, $\mathcal{E}_2^{(n)}$ does not contain 1.

Accordingly, let us consider the case, $n = p^k + 1$. Then $N_{(n,p)} = N_{(p^k,p)}$ and $\mathcal{E}_p^{(p^k)} = 0, \cdots, N_{(p^k,p)}$ except $kp^k - 1$ and except 1 if $p = 2$.

Now, if $k = 1$ and $p = 2$, then $kp^k - 1 = 1$ whence

$$(31) \qquad \mathcal{E}_2^{(3)} = 0, \cdots, N_{(3,2)} \text{ except } 1,$$

* I. Schur, *Mathematische Zeitschrift*, Vol. 29 (1929), pp. 464-465.

and if $k=1$ and $p>2$ we have

(32) $\mathcal{E}_p{}^{(n)}$ contains $\mathcal{E}_p{}^{(p-1)}+\mathcal{E}_p{}^{(2)}$ which contains $p-1$, whence we have, for $n=p^k+1$ and $k=1$,

(33) $$\mathcal{E}_p{}^{(n)}=0,\cdots,N_{(n,p)} \text{ (except 1 if } p=2)$$

which verifies Theorem 8 for the case, $n=p+1$.

Now, if $k>1$ in this same case $(n=p^k+1)$ it may readily be verified that there exist no two rational integers greater than *one*, n' and n'', such that

$$n=n'+n'' \text{ and } N_{(n',p)}+N_{(n'',p)}\geq kp^k-1$$

whence by (30) as $\mathcal{E}_p{}^{(p^k)}$ does not contain kp^k-1 we have in this case

$$H_p{}^{(n)} \text{ does not contain } kp^k-1$$

and by Theorem 4 for $n=p^k+1$ we have $A_p{}^{(n)}$ does not contain kp^k-1 where $k>1$. Therefore, by Theorem 8, restricted, and 1 of Theorem 3 we have, as $N_{(p^k,p)}=N_{(p^k+1,p)}$

(34) $$\text{If } k>1, \text{ then } \mathcal{E}_p{}^{(n)}=\mathcal{E}_p{}^{(p^k)} \text{ if } n=p^k+1.$$

Now, consider the case, $n=bp^k$ where b is a rational integer, and $1<b<p$. Obviously, $p=2$ is excluded from this case. Then by the definition of k and (30) if we set $n'=(b-1)p^k$ and $n''=p^k$ then if $\mathcal{E}_p{}^{(n')}$ is as given in Theorem 8 we have

(35) $$\text{as } n=n'+n'',\ H_p{}^{(n)} \text{ includes } 0,\cdots,(N_{(n',p)}+N_{(n'',p)})$$

but by (3) and (4) we have in this case $N_{(n',p)}+N_{(n',p)}=N_{(n,p)}-1$ whence (35) and Theorem 1 give (if $\mathcal{E}_p{}^{(n')}$ is correctly given by Theorem 8 where $n'=(b-1)p^k$) then

(36) $$\mathcal{E}_p{}^{(n)}=0,\cdots,N_{(n,p)};$$

(where $n=bp^k$ as provided above) which is as given by *Theorem 8.* But for $b=2$, $n'=p^k$; whence by definition of k, $\mathcal{E}_p{}^{(n)}$ is in this case correctly given by Theorem 8; whence by complete induction we have by (36) for any rational integer, $b>1$ and $<p$

(37) $$\text{if } n=bp^k \text{ then } \mathcal{E}_p{}^{(n)}=0,\cdots,N_{(n,p)},$$

which is as given in Theorem 8.

Now consider every other case for $n<p^{k+1}$; i.e., $n>p^k+1$ and such that there exists no positive rational integer, b, such that $n=bp^k$. Then n is given in p-adic form by (3) where $q=k$; then

(38) $$n = \sum_{a=0}^{k} b_a p^a, \quad \text{where} \quad 0 \leq b_a < p$$

and b_a is a rational integer. Obviously, as $n > p^k + 1$ we have $b_k \neq 0$. Now, let $b = b_k$ and $n' = bp^k$. Then in (30) by substitution we have $n'' = n - bp^k$ whence (38) gives

(39) $$n'' = \sum_{a=0}^{k-1} b_a p^a < p^k \quad \text{and, if } b = 1, \text{ then } n'' > 1;$$

whence by (30), (37) and the definition of k we have in this case

(40) $$\mathcal{E}_p^{(n)} = 0, \cdots, N_{(n,p)} \qquad (\text{except 1 if } p = 2).$$

Accordingly, we have shown that if Theorem 8 is verified for $n \leq p^k$, then it can be verified for $n < p^{k+1}$. It remains but to show that then it can be verified for $n = p^{k+1}$.

Consider this case, $n = p^{k+1}$. Then for any two positive rational integers, n' and n'', such that $n = n' + n''$, it can be deduced from the definition of $N_{(n,p)}$ and the relations (3) and (4), by replacement of n by n' and n'' in the argument successively, that in any instance we have

(41) $$\text{setting } k' = k + 1, \quad \text{then} \quad k'p^{k'} - 2 \geq N_{(n',p)} + N_{(n'',p)};$$

and, indeed, that equality exists only when $n' \equiv 0 \bmod p^k$).

Now, let $n' = (p - 1)p^k$. Then $n'' = p^k$, and in (41) we have

(42) $$k'p^{k'} - 2 = N_{(n';p)} + N_{(n'',p)};$$

whence by (41), (30) and the definition of $H_p^{(n)}$ we have by Theorem 8 (restricted to the domain verified)

(43) $$\text{if } n = p^{k'}, \text{ then } H_p^{(n)} = 0, \cdots, (k'p^{k'} - 2), \quad (\text{except 1 if } p = 2).$$

We now turn to a consideration of the components of $A_p^{(n)}$ for $n = p^{k'}$. These are given by Theorems 4 and 5 by reference to (43) by

(44) $$\text{if } n = p^{k'}, \text{ then } A_p^{(n)} = k'p^{k'}, \cdots, N_{(n,p)};$$

whence by (43) we have verified Theorem 8 for the case, $n = p^{k+1}$, which was all that remained to be done in order to establish this theorem completely.

Theorem 8 is stated in another form in the introduction. It may be verified readily that the two statements are equivalent and that every possible case is covered.

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TWO-DIMENSIONAL CHAINS.

By A. Arwin.

In the present paper I have set myself the problem of finding a generalization in two or more dimensions of the usual chains of fractions, which we may briefly speak of as one-dimensional chains, and their periodicity. Of earlier papers on this theme I mention those of Berwick * and Daus,† and above all those of Perron,‡ who has done close investigation of the chains which he calls "Jacobi-chains," and of their convergence, periodicity and order of approximation. The two-dimensional Jacobi-chains do not, however, enjoy the property of periodicity in conjunction with that of the "best possible order of approximation," § as the one-dimensional chains for a quadratic irrationality do. It is therefore of interest to know that it is possible also in the case of cubic irrationalities to form chains of periodicity together with the highest order of approximation by following up a generalization of the one-dimensional chains along lines, which will be explained below. The best order of approximation of two cubic irrationalities μ_1, μ_2 is the one given by

$$(1) \qquad | \mu_1 - x_r/z_r | \leqq k_1/z_r^{3/2}, \quad | \mu_2 - y_r/z_r | \leq k_2/z_r^{3/2},$$

where x, y, z are rational integers, k_1 and k_2 constants $\nrightarrow 0$, greater than a readily assigned numerical quantity.¶

The principle to be applied may briefly be characterized as follows. Let us have the one-dimensional chain, homogeneously written

$$(2) \qquad \omega_0^{(1)} = \omega_{r+1}^{(1)} x_{r+1} + \omega_{r+1}^{(2)} x_r, \quad \omega_0^{(2)} = \omega_{r+1}^{(1)} z_{r+1} + \omega_{r+1}^{(2)} z_r,$$

and, as known, the order of approximation

$$\omega_0^{(1)}/\omega_0^{(2)} - x_r/z_r = \delta_r/z_r^2$$

where $\omega_0^{(i)}$, $\omega_{r+1}^{(i)}$ are quadratic, algebraic integers and the constant δ_r limited, possibly $\rightarrow 0$. On account of this equation the relations (2) will give

* W. E. H. Berwick, *Proceedings of the London Mathematical Society* (2), Vol. 12 (1912).

† P. H. Daus, *American Journal of Mathematics*, Vol. 44 (1922).

‡ O. Perron, *Mathematische Annalen*, Bd. 64 (1907); *Sitzungsberichte der Königlischen Bayerschen Akademie der Wissenschaften zu München*, Bd. 37 and 38.

§ O. Perron, *Irrationalzahlen*, 1921, s. 135.

¶ O. Perron, *Mathematische Annalen*, Bd. 83 (1921).

$$(-1)^{r+1}\omega_{r+1}^{(1)}=\omega_0^{(2)}\delta_r/z_r$$

and for $\bar{\omega}_{r+1}^{(1)}$, the conjugate of $\omega_{r+1}^{(1)}$

$$(-1)^{r+1}\bar{\omega}_{r+1}^{(1)}=\bar{\omega}_0^{(2)}z_r(\bar{\omega}_0^{(1)}/\bar{\omega}_0^{(2)}-\omega_0^{(1)}/\omega_0^{(2)}+\delta_r/z_r^2).$$

With increasing r we have the norm

$$N(\omega_{r+1}^{(1)})\sim\omega_0^{(2)}\bar{\omega}_0^{(2)}(\bar{\omega}_0^{(1)}/\bar{\omega}_0^{(2)}-\omega_0^{(1)}/\omega_0^{(2)})\delta_r.$$

As however $\omega_{r+1}^{(1)}$ is an algebraic integer, its norm is a rational integer and therefore $\nrightarrow 0$. Hence, since the chain cannot break off, $\delta_r \nrightarrow 0$ and $N(\omega_{r+1}^{(1)})$ as well as $N(\omega_{r+1}^{(2)})$ are limited for all r, from which we easily infer that in the one-dimensional chain from a quadratic irrationality the formation is periodic. From this point of view, we are going to investigate the two-dimensional chains formed by cubic irrationalities. Let us therefore take μ_1 and μ_2 from a cubic field and let us assume, as is usually done (a proof of this statement however we omit here) that

$$x-\mu_1 z=0, \qquad y-\mu_2 z=0 \tag{4}$$

form a vector in space. Then we have to determine a set of points $A_r(x_r, y_r, z_r)$ which shall approximate the vector (4). Let us have, for example, constructed $A_r(x_r, y_r, z_r)$ in the shortest distance ρ_r from (4) and h_r from a plane at right angles to (4). Proceeding from A_r with the circular disc $\pi(\rho_r-\epsilon_1)^2$, $\epsilon_1>0$ but otherwise arbitrarily small, we move it along the vector (4) until the next point A_{r+1} with $\rho_{r+1}<\rho_r$ falls on it. In this way we construct a set of points A_r, A_{r+1} etc. with decreasing $\rho_r>\rho_{r+1}>\cdots$. The existence of such a sequence follows from the theorem of Minkowski,* that says: A convex body with a "lattice point" as centre and volume 8 must always without this centre have at least one further lattice point. But we may infer more than only the existence of the set A_r. Then, since no point is given on or within the cylinder $2\pi\rho_r^2 h_{r+1}$, A_r and A_{r+1} excepted, we may contract ρ_r and h_{r+1} arbitrarily little ϵ_1 and ϵ_2, and have no lattice point at all on or within the cylinder $2\pi(\rho_r-\epsilon_1)^2(h_{r+1}-\epsilon_2)$. By reason of this theorem of Minkowski we therefore infer the following important inequality

$$2\pi(\rho_r-\epsilon_1)^2(h_{r+1}-\epsilon_2)<8.$$

$$2\pi\rho_r^2 h_{r+1}<8+4\pi\rho_r h_{r+1}\epsilon_1+2\pi\rho_r^2\epsilon_2+2\pi\epsilon_1^2\epsilon_2-2\pi\epsilon_1^2 h_{r+1}-4\pi\epsilon_1\epsilon_2\rho_r,$$

or, since ϵ_1 and ϵ_2 are arbitrarily small, for example, also

$$2\pi\rho_r^2 h_{r+1}<16, \qquad \rho_r^2 h_{r+1}<8/\pi. \tag{5}$$

* H. Minkowski, *Diophantische Approximationen*, s. 60.

The plane

(6) $$x\mu_1 + y\mu_2 + z = 0$$

is perpendicular to the vector (4), whence h_{r+1} is computed as

(7) $$h_{r+1} = (x_{r+1}\mu_1 + y_{r+1}\mu_2 + z_{r+1})/(\mu_1^2 + \mu_2^2 + 1)^{1/2}$$

and also

(7′) $$\rho_r^2 = R_r^2 \sin^2 \delta_r = \frac{(x_r - \mu_1 z_r)^2 + (y_r - \mu_2 z_r)^2 + (\mu_1 y_r - \mu_2 z_r)^2}{\mu_1^2 + \mu_2^2 + 1}.$$

From our construction of the set A_r it immediately follows that

(8) $$\mu_1 - x_r/z_r = \epsilon_r^{(1)}, \quad \mu_2 - y_r/z_r = \epsilon_r^{(2)}, \quad \epsilon_r^{(1)} \text{ and } \epsilon_r^{(2)} \to 0.$$

Hence we have

$$z_{r+1}\left[1 - \frac{\mu_1\epsilon_{r+1}^{(1)} + \mu_2\epsilon_{r+1}^{(2)}}{\mu_1^2 + \mu_2^2 + 1}\right]\left[(x_r - \mu_1 z_r)^2 + (y_r - \mu_2 z_r)^2 + (\mu_1 y_r - \mu_2 z_r)^2\right] < (8/\pi)(\mu_1^2 + \mu_2^2 + 1)^{1/2}$$

and therefore, for example,

$$(x_r - \mu_1 z_r)^2 < \text{constant}/z_{r+1}, \text{ const.} = (9/\pi)(\mu_1^2 + \mu_2^2 + 1)^{1/2},$$

that is,

(9) $$| \mu_1 - x_r/z_r | < \text{constant}/z_{r+1}.$$

Proceeding as above the set of points A_r will give the following approximations

(9′) $$\mu_1 - x_r/z_r = k_r^{(1)}/z_r^{3/2}, \qquad \mu_2 - y_r/z_r = k_r^{(2)}/z_r^{3/2},$$

where $k_r^{(i)}$ might eventually tend to zero; but, as already said, if μ_1 and μ_2 are cubic, independent irrationalities, Mr. Perron * has proved that actually $k_r^{(i)}$ does not tend to zero but has the order of a numerical constant. From this fact and (9′) we also conclude, and this is important, that in

(10) $$z_{r+1} = \tau_r z_r,$$

τ_r for all r is limited since

$$z_{r+1} < (9/\pi)(\mu_1^2 + \mu_2^2 + 1) z_r/\delta^2,$$

$$\delta \geq 1/2\sigma, \quad \sigma = \prod \rho_i, \quad \rho_i = \sum_{\nu=1}^{2} | \mu_\nu^{(i)} - \mu_\nu | \qquad (i = 1, 2).$$

Let us now put

(11) $$A_{r+1,r} = y_{r+1}z_r - y_r z_{r+1}, \quad B_{r+1,r} = z_{r+1}x_r - z_r x_{r+1}, \quad C_{r+1,r} = x_{r+1}y_r - x_r y_{r+1},$$

and by means of (9′) compute for example

* O. Perron, *Mathematische Annalen*, Bd. 83 (1921).

$$B_{r+1,r}=(z_{r+1}z_r/z_r^{3/2})[-k_r^{(1)}+k_{r+1}^{(1)}(z_r/z_{r+1})^{3/2}],$$

from which we further conclude, that $B_{r+1,r}$ has an order of infinity of at most $z_r^{1/2}$ and in the same way all $A_{m,n}$, $B_{m,n}$, $C_{m,n}$ $(m,n=r,r+1$ and $r+2)$. Hence we write

$$(12)\quad A_{r+1,r}=x_{r+1,r}^{(1)}z_r^{1/2},\quad B_{r+1,r}=x_{r+1,r}^{(2)}z_r^{1/2},\quad C_{r+1,r}=x_{r+1,r}^{(3)}z_r^{1/2},$$

where now all $x_{m,n}^{(i)}$ are limited, eventually $\to 0$. Let us further set

$$(13)\qquad \mu_1=(\omega_0^{(1)}/\omega_0^{(3)}),\quad \mu_2=(\omega_0^{(2)}/\omega_0^{(3)}),$$

where $\omega_0^{(i)}$ are algebraic integers, form with x_r, y_r, z_r the expressions

$$(14)\qquad \begin{aligned}\omega_0^{(1)}&=x_{r+2}\omega_r^{(1)}+x_{r+1}\omega_r^{(2)}+x_r\omega_r^{(3)},\\ \omega_0^{(2)}&=y_{r+2}\omega_r^{(1)}+y_{r+1}\omega_r^{(2)}+y_r\omega_r^{(3)},\\ \omega_0^{(3)}&=z_{r+2}\omega_r^{(1)}+z_{r+1}\omega_r^{(2)}+z_r\omega_r^{(3)},\end{aligned}$$

and solve with regard to $\omega_r^{(i)}$

$$(14')\qquad \begin{aligned}\Delta_{r+2}\omega_r^{(1)}&=A_{r+1,r}\omega_0^{(1)}+B_{r+1,r}\omega_0^{(2)}+C_{r+1,r}\omega_0^{(3)},\\ -\Delta_{r+2}\omega_r^{(2)}&=A_{r+2,r}\omega_0^{(1)}+B_{r+2,r}\omega_0^{(2)}+C_{r+2,r}\omega_0^{(3)},\\ \Delta_{r+2}\omega_r^{(3)}&=A_{r+1,r+1}\omega_0^{(1)}+B_{r+1,r+1}\omega_0^{(2)}+C_{r+1,r+1}\omega_0^{(3)}.\end{aligned}$$

Let us set

$$(14'')\qquad \omega_0^{(i)}=a_0^{(i)}\omega_1+a_1^{(i)}\omega_2+a_2^{(i)}\qquad (i=1,2,3)$$

1, ω_1 and ω_2 forming a base in the cubic field; $\Delta_{r+2}\omega_r^{(i)}$ are from (14′) also algebraic integers. By (9′) and (12) the first relation (14′) yields

$$\begin{aligned}(15)\quad \Delta_{r+2}\omega_r^{(1)}&=(\omega_0^{(3)}/z_r)[x_r(y_{r+1}z_r-y_rz_{r+1})+y_r(z_{r+1}x_r-x_{r+1}z_r)\\ &\quad+z_r(x_{r+1}y_r-x_ry_{r+1})+k_r^{(1)}x_{r+1,r}^{(1)}+k_r^{(2)}x_{r+1,r}^{(2)}+x_{r+1,r}^{(3)}]\\ &=(\omega_0^{(3)}/z_r)[k_r^{(1)}x_{r+1,r}^{(1)}+k_r^{(2)}x_{r+1,r}^{(2)}+x_{r+1,r}^{(3)}].\end{aligned}$$

From this relation we infer that $\Delta_{r+2}\omega_r^{(1)}$ is for increasing r at least of the order $1/z_r$, or possibly of a less order $1/z_r^{1+\epsilon}$, $\epsilon>0$. Let us further with (12) and (14″) form the conjugate integers $\Delta_{r+2}\overline{\omega}_r^{(1)}$ and $\Delta_{r+2}\overline{\overline{\omega}}_r^{(1)}$. We have for example

$$\begin{aligned}(15)\quad \Delta_{r+2}\overline{\omega}_r^{(1)}&=z_r^{1/2}[(a_0^{(1)}x_{r+1,r}^{(1)}+a_0^{(2)}x_{r+1,r}^{(2)}+a_0^{(3)}x_{r+1,r}^{(3)})\overline{\omega}_1\\ &\quad+\overline{\omega}_2(a_1^{(1)}x_{r+1,r}^{(1)}+a_1^{(2)}x_{r+1,r}^{(2)}+a_1^{(3)}x_{r+1,r}^{(3)})\\ &\quad+(a_2^{(1)}x_{r+1,r}^{(1)}+a_2^{(2)}x_{r+1,r}^{(2)}+a_2^{(3)}x_{r+1,r}^{(3)})],\end{aligned}$$

and a similar expression in $\Delta_{r+2}\overline{\overline{\omega}}_r^{(1)}$, from which we infer that these conjugate values are of an order of infinity of at most $z_r^{1/2}$ each. Hence the norm $N(\Delta_{r+2}\omega_r^{(1)})$ must either tend to zero or rest limited, different from zero. In the same way we have $N(\Delta_{r+2}\omega_r^{(2)})$ zero or limited and finally

$N(\Delta_{r+2}\omega_r^{(3)})$. We shall in the following deduce an upper limit for Δ_{r+2}. Since however $\Delta_{r+2}\omega_r^{(i)}$ are algebraic integers the formation of A_r would in the former case go to an end, and this must occur for quadratic irrationalities $\omega_0^{(i)}$ with the order of approximation (9′), but it presumes that a linear relation between the initial elements $\omega_0^{(i)}$ is existing. In general the latter must therefore occur, whence the two expressions (15″) must have the order of infinity of precisely $z_r^{1/2}$ as (15′) of precisely $1/z_r$. Hence the above relation must lead to a set of algebraic integers of limited norm just as formerly in the one-dimensional case and quadratic irrationalities. In two dimensions, however, the periodicity is not yet proved. But, writing

$$\text{(16)}\qquad \Omega_0^{(i)} = \omega_0^{(i)}/\omega_0^{(3)}, \quad \Omega_r^{(i)} = \Delta_{r+2}\omega_r^{(i)}/\Delta_{r+2}\omega_r^{(3)} \qquad (i = 1, 2).$$

whence

$$\text{(17)}\quad \Omega_0^{(1)} = \frac{x_r + x_{r+1}\Omega_r^{(2)} + x_{r+2}\Omega_r^{(1)}}{z_r + z_{r+1}\Omega_r^{(2)} + z_{r+2}\Omega_r^{(1)}}, \quad \Omega_0^{(2)} = \frac{y_r + y_{r+1}\Omega_r^{(2)} + y_{r+2}\Omega_r^{(1)}}{z_r + z_{r+1}\Omega_r^{(2)} + z_{r+2}\Omega_r^{(1)}}$$

we see, since $\Delta_{r+2}\omega_r^{(i)}$ $(i = 1, 2, \text{ and } 3)$ are just proved to have the precise order $1/z_r$, that in

$$\text{(16′)}\qquad \Omega_r^{(i)} = \frac{\alpha_r^{(i)}\omega_1 + s_r^{(i)}\omega_2 + j_r^{(i)}}{R_r},$$

not only $R_r < M$ and $N(\alpha_r^{(i)}\omega_1 + s_r^{(i)}\omega_2 + j_r^{(i)}) < M$, but $\Omega_r^{(i)}$ itself must be limited and hence also the expression $\alpha_r^{(i)}\omega_1 + \beta_r^{(i)}\omega_2 + j_r^{(i)}$. From this fact we are able to conclude that the formation must become periodic. We write namely for

$$\omega_1 = \frac{a_0^{(1)}\omega^2 + b_0^{(1)}\omega + c_0}{R_0^{(1)}}, \qquad \omega_2 = \frac{a_0^{(2)}\omega^2 + b_0^{(2)}\omega + c_0^{(2)}}{R_0^{(2)}}$$

the numerators of $\Omega_r^{(i)}$ after multiplication with $R_0^{(1)}R_0^{(2)}$

$$a_r\omega^2 + b_r\omega + c_r = \eta_r^{(1)}$$
$$a_r\overline{\omega}^2 + b_r\overline{\omega} + c_r = \eta^{(2)}$$
$$a_r\overline{\overline{\omega}}^2 + b_r\overline{\overline{\omega}} + c_r = \eta_r^{(3)}$$

where a_r, b_r, c_r are rational integers, $\eta_r^{(i)}$ limited for all r, and compute, since $\omega + \overline{\omega} + \overline{\overline{\omega}} = -A_1$, $\omega^3 + A_1\omega^2 + A_2\omega + A_3 = 0$, three relations

$$-A_1 - \omega + \frac{b_r}{a_r} = \frac{\eta_r^{(2)} - \eta_r^{(3)}}{a_r(\overline{\omega} - \overline{\overline{\omega}})}.$$

But these three relations are in contradiction if $|a_r| \to \infty$. Hence a_r, b_r, c_r are limited, and the formation periodic. But it is remarkable that the same elements need not follow in the same order in the set of periods; that is,

the periods need not hold the same elements in the same order. The periodicity can be mixed. If $\omega_0^{(i)}$ belong to a field higher than the cubic we get questions and problems almost as in the case of cubic and higher irrationalities in the one-dimensional chains, questions as these, if R_r in (16′) be limited, if $\Omega_r^{(i)}$ themselves are limited, etc. We shall now prove that Δ_r for all r is limited. From

$$\Delta_{r+2} = \begin{vmatrix} x_{r+2} & x_{r+1} & x_r \\ y_{r+2} & y_{r+1} & y_r \\ z_{r+2} & z_{r+1} & z_r \end{vmatrix}, \quad \mu_1 - x_r/z_r = k_r^{(1)}/z_r^{3/2}, \quad \mu_2 - y_r/z_r = k_r^{(1)}/z_r^{3/2}$$

where $k_r^{(i)}$ and τ_r are limited $\nrightarrow 0$ for increasing r, we infer

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 z_r^3 \begin{vmatrix} \mu_1 - k_{r+2}^{(1)}/z_{r+2}^{3/2}, & \mu_1 - k_{r+1}^{(1)}/z_{r+1}^{3/2}, & \mu_1 - k_r^{(1)}/z_r^{3/2} \\ \mu_2 - k_{r+2}^{(2)}/z_{r+2}^{3/2}, & \mu_2 - k_{r+1}^{(2)}/z_{r+1}^{3/2}, & \mu_2 - k_r^{(2)}/z_r^{3/2} \\ 1, & 1, & 1 \end{vmatrix},$$

and further

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 \begin{vmatrix} \phi_{r+1}^{(1)}, & \phi_r^{(1)}, & (\mu_1 - k_r^{(1)}/z_r^{3/2})z_r^{3/2} \\ \phi_{r+1}^{(2)}, & \phi_r^{(2)}, & (\mu_2 - k_r^{(2)}/z_r^{3/2})z_r^{3/2} \\ 0, & 0, & 1 \end{vmatrix},$$

that is in

$$\Delta_{r+2} = \tau_{r+1}\tau_r^2 \begin{vmatrix} \phi_{r+1}^{(1)}, & \phi_r^{(1)} \\ \phi_{r+1}^{(2)}, & \phi_r^{(2)} \end{vmatrix}$$

all elements limited. Hence an upper limit of Δ_{r+2} is readily deduced. By means of the above geometrical considerations it is yet in general possible to limit Δ_{r+2} to the values of some few of the smallest integers.

Let us then propose any relation

$$x_{r+3} = \alpha x_{r+2} + \beta x_{r+1} + \gamma x_r$$

given in rational fractions α, β and γ with

$$(19') \qquad \begin{aligned} \begin{vmatrix} x_{r+3} & y_{r+2} & z_{r+1} \end{vmatrix} &= \Delta_{r+3}, \\ \begin{vmatrix} x_{r+2} & y_{r+1} & z_r \end{vmatrix} &= \Delta_{r+2}. \end{aligned}$$

Then we compute $\gamma = \Delta_{r+3}/\Delta_{r+2}$ and hence the recursion formula

$$(19'') \qquad \Delta_{r+2}x_{r+3} = m_{r+3}x_{r+2} + n_{r+3}x_{r+1} + \Delta_{r+3}x_r,$$

for if x'_{r+3}, y'_{r+3}, z'_{r+3} be any solutions of (19′) we can always determine m_{r+3} and n_{r+3} as integers from (19″), whence this formula must afford all solutions of (19′). Let us then substitute $r+1$ for r in (14′), whence the relations

$$
(20)\qquad
\begin{aligned}
&\omega_r^{(1)}x_{r+2}+\omega_r^{(2)}x_{r+1}+\omega_r^{(3)}x_r=\omega_{r+1}^{(1)}x_{r+3}+\omega_{r+1}^{(2)}x_{r+2}+\omega_{r+1}^{(3)}x_r,\\
&\omega_r^{(1)}y_{r+2}+\omega_r^{(2)}y_{r+1}+\omega_r^{(3)}y_r=\omega_{r+1}^{(1)}y_{r+3}+\omega_{r+1}^{(2)}y_{r+2}+\omega_{r+1}^{(3)}y_r,\\
&\omega_r^{(1)}z_{r+2}+\omega_r^{(2)}z_{r+1}+\omega_r^{(3)}z_r\ +\omega_{r+1}^{(1)}z_{r+3}+\omega_{r+1}^{(2)}z_{r+2}+\omega_{r+1}^{(3)}z_r.
\end{aligned}
$$

Solving this system with respect to $\omega_{r+1}^{(i)}$ we have proved the following relations

$$
(21)\qquad
\begin{aligned}
\Delta_{r+3}\omega_{r+1}^{(1)}&=\Delta_{r+2}\omega_r^{(3)},\\
\Delta_{r+3}\Delta_{r+2}\omega_{r+1}^{(2)}&=-\Delta_{r+2}m_{r+3}\omega_r^{(3)}+\Delta_{r+3}\Delta_{r+2}\omega_r^{(1)},\\
\Delta_{r+3}\Delta_{r+2}\omega_{r+1}^{(3)}&=-\Delta_{r+2}n_{r+3}\omega_r^{(3)}+\Delta_{r+3}\Delta_{r+2}\omega_r^{(2)},
\end{aligned}
$$

also written in form

$$
(21')\qquad
\begin{aligned}
\Delta_{r+3}\Omega_r^{(1)}&=m_{r+3}+\Delta_{r+2}(\Omega_{r+1}^{(2)}/\Omega_{r+1}^{(1)}),\\
\Delta_{r+3}\Omega_r^{(2)}&=n_{r+3}+\Delta_{r+2}(1/\Omega_{r+1}^{(1)}),
\end{aligned}
$$

which suggests that we search for m_{r+3} and n_{r+3} as characteristic integers in $\Omega_r^{(i)}$. Consider the plane

$$
xA_{r+2,r+1}+yB_{r+2,r+1}+zC_{r+2,r+1}=\Delta_{r+3}
$$

carrying all points x_{r+3}, y_{r+3}, z_{r+3}, and determine the intersection of this plane with the vector (4). We find the coordinates

$$
(22)\qquad
\begin{aligned}
x^0{}_{r+3}&=\frac{\mu_1\Delta_{r+3}}{A_{r+2,r+1}\mu_1+B_{r+2,r+1}\mu_2+C_{r+2,r+1}},\\
y^0{}_{r+3}&=\frac{\mu_2\Delta_{r+3}}{A_{r+2,r+1}\mu_1+B_{r+2,r+1}\mu_2+C_{r+2,r+1}},\\
z^0{}_{r+3}&=\frac{1}{A_{r+2,r+1}\mu_1+B_{r+2,r+1}\mu_2+C_{r+2,r+1}},
\end{aligned}
$$

and by varying m_{r+3}, n_{r+3} so as to make (22) satisfy (19″) we compute further

$$
\begin{aligned}
m^0{}_{r+3}&=-\Delta_{r+3}\frac{A_{r+1,r}\mu_1+B_{r+1,r}\mu_2+C_{r+1,r}}{A_{r+2,r+1}\mu_1+B_{r+2,r+1}\mu_2+C_{r+2,r+1}}=\Delta_{r+3}\frac{\omega_r^{(1)}}{\omega_r^{(3)}}=\Delta_{r+3}\Omega_r^{(1)},\\
n^0{}_{r+3}&=-\Delta_{r+3}\frac{A_{r+2,r}\mu_1+B_{r+2,r}\mu_2+C_{r+2,r}}{A_{r+2,r+1}\mu_1+B_{r+2,r+1}\mu_2+C_{r+2,r+1}}=\Delta_{r+3}\frac{\omega_r^{(2)}}{\omega_r^{(3)}}=\Delta_{r+3}\Omega_r^{(2)}.
\end{aligned}
$$

$m^0{}_{r+3}$ and $n^0{}_{r+3}$ are easy to characterize as the way that we have to proceed along the edges of a tetrahedron in order to reach just the point $\Delta_{r+2}\, x^0{}_{r+3}$, $y^0{}_{r+3}$, $z^0{}_{r+3}$ and consequently m_{r+3} and n_{r+3} are rational integers in $m^0{}_{r+3}$, $n^0{}_{r+3}$, which lead to the "best" approximating point x_{r+3}, y_{r+3}, z_{r+3} close by $x^0{}_{r+3}$, $y^0{}_{r+3}$, $z^0{}_{r+3}$. To determine the integers m_{r+3} and n_{r+3} as well as Δ_r we have to proceed from $\Delta_{r+3}=1$, decide, as above explained, on the "best" integers in $\Omega_r^{(1)}$ and $\Omega_r^{(2)}$, compute x_{r+3}, y_{r+3}, z_{r+3} and test, if $\rho_{r+2}>\rho_{r+3}$ is true.

If so, the point A_{r+3} is found; if not, we have to repeat this process with $\Delta_{r+3}=2$, 3 etc. until A_{r+3} will be determined; but we have no algorithm. which gives us m_r and n_r automatically, as in the case of the one-dimensional chains. As a special case of this mode of forming chains we may consider the "Jacobi-chains," in which without any exception Δ_r is equal to one, and m_{r+3}, n_{r+3} are determined as greatest integers in $\Omega_r^{(1)}$ and $\Omega_r^{(2)}$. But, what in this manner is won in simplicity, is a loss in generality, namely loss of periodicity in connection with the best order of approximation, and of this simple proof of convergence. Hence, in order to attain the generalization of the one-dimensional chains to two dimensions we have to release the conditions $\Delta_r=1$ and permit mixed periodicity. As already indicated, a formation with any quadratic irrationalities must break off, and this is readily proved directly also with $k_r^{(i)}\to 0$. For if $\omega_0^{(i)}$ are quadratic irrationalities there exists in integers a_0, b_0, c_0 a relation

$$a_0(\omega_0^{(1)}/\omega_0^{(3)})+b_0(\omega_0^{(2)}/\omega_0^{(3)})+c_0=0$$

from which by the approximation (9) it follows that

$$a_0x_r+b_0y_r+c_0z_r\to 0\,;$$

that is

$$a_0x_r+b_0y_r+c_0z_r=0$$

for $n>N$, which is impossible. But as the Jacobi-chains in two dimensions also possess the same property,* and as they are easy to form, we shall use them for the purpose of solving in general a problem from the theory of numbers. It is however of some interest to show that also with quadratic elements a periodic chain is possible to construct in the following simple way.

$$7+3(7)^{1/2}=14+\cfrac{1}{\cfrac{3(7)^{1/2}+7}{14}}; \qquad 5+2(7)^{1/2}=10+\cfrac{\cfrac{7-(7)^{1/2}}{14}}{\cfrac{3(7)^{1/2}+7}{14}}$$

$$\frac{3(7)^{1/2}+7}{14}=1+\cfrac{1}{\cfrac{3(7)^{1/2}+7}{1}}; \qquad \frac{7-(7)^{1/2}}{14}=0+\cfrac{\cfrac{(7)^{1/2}+2}{1}}{\cfrac{3(7)^{1/2}+7}{1}}$$

$$\frac{3(7)^{1/2}+7}{1}=14+\cfrac{1}{\cfrac{3(7)^{1/2}+7}{14}}; \qquad \frac{(7)^{1/2}+2}{1}=4+\cfrac{\cfrac{7+(7)^{1/2}}{14}}{\cfrac{3(7)^{1/2}+7}{14}}$$

* O. Perron, *Sitzungsberichte der Königlichen Bayerschen Akademie der Wissenschaften zu München*, Bd. 38 (1908).

$$\frac{3(7)^{\frac{1}{2}}+7}{14} = 1 + \cfrac{1}{\cfrac{3(7)^{\frac{1}{2}}+7}{1}}; \qquad \frac{7+(7)^{\frac{1}{2}}}{14} = 0 + \cfrac{\cfrac{5+2(7)^{\frac{1}{2}}}{1}}{\cfrac{3(7)^{\frac{1}{2}}+7}{1}}.$$

We shall now treat the named problem of solving any ternary, quadratic form in a quadratic field. Let us therefore have the general ternary form

$$f(xyz) = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$$

$$= ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy,$$

and let us assume that by means of any substitution

$$(23)\qquad \begin{aligned} x &= \alpha_0 t + \alpha_1 u, \\ y &= \beta_0 t + \beta_1 u, \\ z &= \gamma_0 t + \gamma_1 u, \end{aligned}$$

the binary quadratic form $(p, q, r) = \psi(t, u)$ is brought into the quality

$$(24)\qquad f(x, y, z) = \psi(t, u).$$

This problem is treated already by Gauss. Putting the root of $\psi(t, u) = 0$ in (24) we have from (23) to each substitution a diophantine solution of

$$(25)\qquad f(x^0y^0z^0) = 0$$

in the quadratic field $K[(d)^{\frac{1}{2}}]$, s^2d or $d = p^2 - qr$, and we observe that the solutions of (25) in $K[(d)^{\frac{1}{2}}]$ can, when existing, be arranged after classes of forms in $K[(d)^{\frac{1}{2}}]$. Now it is interesting that conversely from each solution of (25) it is possible to derive a substitution (23). For, having found the solutions $\mu = x^0/z^0$, $\nu = y^0/z^0$, where x^0, y^0, z^0 are algebraic integers in $K[(d)^{\frac{1}{2}}]$, we form as above the Jacobi-chains, and from the stopping chain we shall see, that a substitution (23) is readily constructed. Hence: The necessary and sufficient condition that any ternary, quadratic form may have diophantine solutions in a quadratic field $K[(d)^{\frac{1}{2}}]$ is the existence of relations (24), which give all solutions. Let us have the ternary form

$$(26)\qquad x^2 + 2y^2 + 3z^2 + 4yz + 4xz + 3xy$$

and seek for example solutions in $K[(7)^{\frac{1}{2}}]$. For $\mu = x/z$, $\nu = y/z$ we find a pair of solutions from

$$\mu = -\frac{3\nu+4}{2} \pm \frac{(\nu^2+8\nu+4)^{\frac{1}{2}}}{2}, \qquad (\nu^2+8\nu+4 = a^2)$$

$$\nu = 4 \pm (12 + a^2)^{\frac{1}{2}},$$

with, for example, $a=4$, yielding $-\mu=3(7)^{1/2}-6$, $\nu=2(7)^{1/2}-4$. From them we form the following chain

$$\nu=2(7)^{1/2}-4=1+\frac{1}{[2(7)^{1/2}+5]/3=\alpha_1{}^{(1)}};$$

$$-\mu=3(7)^{1/2}-6=1+\frac{[7+(7)^{1/2}]/3=\alpha_1{}^{(2)}}{[2(7)^{1/2}+5]/3};$$

$$\frac{7+(7)^{1/2}}{3}=3+\frac{1}{[(7)^{1/2}+2]/1=\alpha_2{}^{(1)}};$$

$$\frac{2(7)^{1/2}+5}{3}=3+\frac{2=\alpha_2{}^{(2)}}{[(7)^{1/2}+2]/1};$$

and the chain is breaking off, as it should, already with $\alpha_2{}^{(2)}=2$ falling in $K(1)$. Hence we compute the following equivalences

$$\nu=\frac{\alpha_2{}^{(2)}+4\alpha_2{}^{(1)}}{\alpha_2{}^{(2)}+3\alpha_2{}^{(1)}},\qquad -\mu=\frac{1+\alpha_2{}^{(2)}+6\alpha_2{}^{(1)}}{\alpha_2{}^{(2)}+3\alpha_2{}^{(1)}}.$$

Returning to homogeneous coordinates $\mu=x/z$, $\nu=y/z$, $\alpha_2{}^{(2)}=x'/z'$, $\alpha_2{}^{(1)}=y'/z'$ the unimodular substitution

$$\begin{aligned} -x&=x'+6y'+1z'\\ y&=x'+4y'+0z'\\ z&=x'+3y'+0z' \end{aligned} \tag{27}$$

transforms (26) into the equivalent form

$$3x'^2-y'^2+z'^2-12z'y'-5z'x'+8x'y',$$

and by means of the substitution $x'=2t$, $y'=u$, $z'=t$ this form is transformed in the binary form

$$-u^2+4tu+3t^2 \tag{28}$$

having just the root $\alpha_2{}^{(1)}=2+(7)^{1/2}$ from the chain above. Hence the substitution

$$\begin{aligned} -x&=3t+6u\\ y&=2t+4u\\ z&=2t+3u \end{aligned} \tag{29}$$

formed by the chain lead directly from (26) to (28). The transformations of (28) into itself, and the transformations of it into equivalent forms gives rise to further solutions; and we see, how all solutions of (26) must be reached.

I will finally simply sketch the following generalization in the four

dimensional-space $(xyzt)$. The definition of an angle ω between two lines is, as in euclidean space, also here given by

$$\cos\omega = \sum \cos\delta_i \cos\phi_i,$$

where δ_i and ϕ_i represent the direction angles of the two lines with the axes x, y, z, t. Any "plane" has the equation

$$A_1x + A_2y + A_3z + A_4t + A_5 = 0,$$

and cuts a three-dimensional space out of $(xyzt)$. A three-dimensional cube can therefore, for example, exist in this "plane." Its distance from the point x_1, y_1, z_1, t_1 is

$$d = \pm \frac{A_1x_1 + A_2y_1 + A_3z_1 + A_4t_1 + A_5}{[\sum_1^4 A_i^2]^{1/2}}.$$

After these preliminaries we assume μ_1, μ_2 and μ_3 to belong to the same algebraic field and let

$$x - \mu_1 t = 0, \quad y - \mu_2 t = 0, \quad z - \mu_3 t = 0 \tag{30}$$

represent a vector in $(xyzt)$. A plane perpendicular to this vector at the vertex has the equation

$$x\mu_1 + y\mu_2 + z\mu_3 + t = 0. \tag{31}$$

Let further $\alpha_n^{(i)}$ $(i = 1, 2, 3, 4)$ represent any lattice point; then we compute the cosine between (30) and the line from the vertex to $\alpha_n^{(i)}$ as

$$\cos\omega_n = \frac{\sum \mu_i \alpha_n^{(i)}}{[\sum \mu_i^2 \cdot \sum \alpha_n^{(i)2}]^{1/2}}, \qquad \mu_4 = 1, \tag{32}$$

that is

$$\sin^2\omega_n = \frac{\sum \mu_i^2 \sum \alpha_n^{(i)2} - [\sum \mu_i \alpha_n^{(i)}]^2}{[\sum \mu_i^2 \sum \alpha_n^{(i)2}]} =$$

$$\frac{\sum_{i=1}^{3} (\alpha_n^{(i)} - \mu_i \alpha_n^{(4)})^2 + \sum_{\tau \neq i = 1}^{3} (\mu_\tau \alpha_n^{(i)} - \mu_i \alpha_n^{(\tau)})^2}{\sum \mu_i^2 \sum \alpha_n^{(i)2}}$$

and the length d_{n+1} from $\alpha_{n+1}^{(i)}$ $(i = 1, 2, 3, 4)$ to (31)

$$d_{n+1} = \frac{\sum_1^4 \mu_i \alpha_{n+1}^{(i)}}{[\sum \mu_i^2]^{1/2}}. \tag{33}$$

Any plane (31′) parallel to (31) has a point common with (30), and around this point in (31′) we lay, as we have already laid a circle in the plane of

the three-dimensional space, a sphere of radius r_n and of volume $4/3\pi r_n^3$. Also here we let this sphere glide along the vector and get a convex figure, by which the above theorem of Minkowski is to be proved with a constant 2^4 instead of the above 2^3. We then deduce the inequality

$$(4/3)\pi d_{n+1} r_n^3 < \text{constant}$$

where

$$r_n = R_n \sin \omega_n, \quad R_n = [\sum \alpha_n^{(i)2}]^{1/2}, \quad d_{n+1} = \frac{\sum \mu_i \alpha_{n+1}^{(i)}}{[\sum \mu_i^2]^{1/2}}.$$

From this our way of constructing new lattice points it follows that

$$\alpha_\tau^{(i)}/\alpha_\tau^4 - \mu_i = \epsilon_\tau^{(i)}, \quad \epsilon_\tau^{(i)} \to 0, \quad \tau \to \infty \qquad (i = 1, 2, 3).$$

Hence, as above

$$[\sum_{i=1}^{3} (\alpha_n^{(i)} - \mu_i \alpha_n^{(4)})^2 + \sum_{\tau \neq i = 1}^{3} (\alpha_n^{(i)} \mu_\tau - \alpha_n^{(\tau)} \mu_i)^2]^{3/2} < k_1/\alpha_{n+1}^{(4)},$$

and, since all six terms are positive, so much the more

$$| \alpha_n^{(i)} - \mu_i \alpha_n^{(4)} | < k_2^{(i)}/\alpha_{n+1}^{(4)1/3} \qquad (i = 1, 2, 3)$$

and because of the theorem of Mr. Perron $\alpha_{n+1}^{(4)} = \tau_n^{(4)} \alpha_n^{(4)}$, that is

$$\alpha_n^{(i)}/\alpha_n^{(4)} - \mu_i = k_n^{(i)}/\alpha_n^{(4)} \alpha_n^{(4)1/3} \qquad (i = 1, 2, 3) \tag{34}$$

$\tau_n^{(4)}$ and $k_n^{(i)}$ limited, and this is exactly our fundamental formula, from which other data follow. In the same way we may also give the generalization to any higher dimension than four.

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Malmö, Sweden.

TENSORS OF THE CALCULUS OF VARIATIONS.

By Marie M. Johnson.

In this paper tensors are discussed which are connected with the non-parametric problem of the calculus of variations. The integral

$$J = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\, dx$$

is taken along arcs whose equations are $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n)$ and which join two fixed points in the $(x, y_1, \cdots, y_n)$ space.

For the parametric problem with the integral

$$J = \int_{u_1}^{u_2} F(y_1, \cdots, y_n, y_1', \cdots, y_n')\, du$$

Murnaghan * has shown that the functions $F_{y_i'}$ form a covariant tensor of rank 1 and that there is a contravariant tensor associated with the equations of the geodesics. It is also possible for the parametric problem to prove that the expressions in Euler's differential equations form a covariant tensor. The Weierstrass E-function, the quadratic form used in the Legendre condition, and the expression in the transversality condition are all invariants.†

In the following pages it is shown that when the non-parametric case is considered the $n + 1$ functions $(F - y_1'F_{y_1'} - \cdots - y_n'F_{y_n'})$, $F_{y_i'}$, instead of the n functions $F_{y_i'}$ of the parametric case, are the components of a covariant tensor. Likewise a function has to be added to the n expressions in Euler's differential equations to form a covariant tensor. An application is made of this latter tensor to deduce very simply the relations between canonical differential equations and their transforms by a canonical transformation of coördinates.‡ It is found further that the expression in the transversality condition is an invariant, while the quadratic form of the Legendre condition and the Weierstrass E-function transform so that a factor is introduced. In addition to these results it turns out that there is a covariant tensor of rank 1 which is connected with Jacobi's differential equations. Furthermore the laws of transformation of the two determinants which are used to find the conjugate points of Jacobi's condition are discussed.

* F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, pp. 86-90.

† G. A. Bliss, *Lecture Notes*, Autumn 1926.

‡ See for example, the chapter by Carathéodory in Riemann-Weber, *Die Differential- und Integralgleichungen der Mechanik und Physik*, Teil 1 (1925), pp. 201-205.

1. *Preliminary notions.* In this section we shall describe the integral of the non-parametric problem of the calculus of variations and the transformations of coördinates to which the necessary conditions of this problem are to be subjected. The laws of transformation of tensors in non-parametric space are also stated.

In the non-parametric problem of the calculus of variations the integral to be minimized has the form

$$J = \int_{x_1}^{x_2} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n')\,dx \equiv \int_{x_1}^{x_2} F(x, y, y')\,dx. \tag{1}$$

The symbols y and y' stand for the sets $(y_1, \cdots, y_n)$ and $(y_1', \cdots, y_n')$, respectively, and the primes indicate derivatives with respect to x. The integral is taken along arcs E_{12} which join two fixed points 1 and 2 and whose equations are

$$y_i = y_i(x) \quad (x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n). \tag{2}$$

Let $u, v_1, \cdots, v_n$ be new coördinates for which u is the independent variable. The two systems of coördinates are related by means of a point transformation

$$x = x(u, v_1, \cdots, v_n) \equiv x(u, v), \quad y_i = y_i(u, v_1, \cdots, v_n) \equiv y_i(u, v) \quad (i = 1, \cdots, n), \tag{3}$$

whose Jacobian

$$\mathcal{J} \equiv \partial(x, y)/\partial(u, v) \tag{4}$$

is different from zero in the region of the $(n+1)$-dimensional space under discussion. The transformation sets up a one-to-one correspondence between the points of a region of the (x, y) space and the points of the corresponding region of the (u, v) space. Also it possesses all the continuity properties that are needed in the following arguments. By a transformation (3) the derivatives $y_i', y_i'', \cdots$ with respect to x along an arc in the (x, y) space can be expressed in terms of the derivatives with regard to $u, v_i', v_i'', \cdots$, taken along the corresponding arc in the (u, v) space. This gives the equations

$$y_i' = \frac{\dfrac{\partial y_i}{\partial u} + \dfrac{\partial y_i}{\partial v_a} v_a'}{\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v_a} v_a'} \qquad (i = 1, \cdots, n), \tag{5}$$

and further similar formulas for y_i'', etc.

Whenever a Greek subscript or superscript occurs twice in a term, it is intended that a sum of terms shall be represented. The sum is procured by

letting the index take on the values $1, \cdots, n$ and by summing the resulting expressions.

The inverse of transformation (3) and the solutions of equations (5) for v_i' are given by the following equations:

$$u = u(x, y_1, \cdots, y_n) \equiv u(x, y), \qquad v_i = v_i(x, y_1, \cdots, y_n) \equiv v_i(x, y),$$

$$(6) \qquad v_i' = \frac{\dfrac{\partial v_i}{\partial x} + \dfrac{\partial v_i}{\partial y_a} y_a'}{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y_a} y_a'} \qquad (i = 1, \cdots, n).$$

The Jacobian of this transformation from (x, y) to (u, v) coördinates is the reciprocal of the determinant in (4).

Every arc (2) in the (x, y) space has in the (u, v) space a corresponding arc whose equations are found as follows. By substituting equations (2) in the first equation of (6) we obtain

$$(7) \qquad u = u(x, y(x)).$$

In order to be able to solve for x we make the assumption that $du/dx \neq 0$ along the arcs considered. This is fundamental throughout the succeeding discussions. Let the solution for x of equation (7) be given by $x = x(u)$. By means of this result and equations (2) the first group of n equations in (6) becomes $v_i = v_i(u)$ ($u_1 \leqq u \leqq u_2$; $i = 1, \cdots, n$) which are the equations of the corresponding arc in the (u, v) space. On this arc the images of the points defined by x_1 and x_2 are determined by values u_1 and u_2 which are found by putting x_1 and x_2 in equation (7).

We will now define tensors whose components in the (x, y) space are functions of x, y_i and the successive derivatives y_i', y_i'', $\cdots$ of y_i with regard to x. Suppose that a set of $n + 1$ functions $A_0(x, y, y', \cdots)$, $A_i(x, y, y', \cdots)$ is transformed by every transformation (3) and associated equations (5) into a new set $a_0(u, v, v', \cdots)$, $a_i(u, v, v', \cdots)$ ($i = 1, \cdots, n$) in such a way that

$$a_0 = A_0 \partial x/\partial u + A_a \partial y_a/\partial u, \qquad a_i = A_0 \partial x/\partial v_i + A_a \partial y_a/\partial v_i \qquad (i = 1, \cdots, n).$$

Then the functions A_0, A_i are the components in the (x, y) space of a covariant tensor of rank 1 and the functions a_0, a_i are the components of the same tensor in the (u, v) space. If $n + 1$ functions $A^0(x, y, y', \cdots)$, $A^i(x, y, y', \cdots)$ are transformed so that

$$a^0 = A^0 \partial u/\partial x + A^a \partial u/\partial y_a, \qquad a^i = A^0 \partial v_i/\partial x + A^a \partial v_i/\partial y_a \qquad (i = 1, \cdots, n),$$

then A^0, A^i are the components of a contravariant tensor of rank 1. These definitions can be extended readily to those of tensors of higher rank.*

2. *Tensors associated with Euler's differential equations.* The non-parametric problem of the calculus of variations in the (x, y) space is transformed to the (u, v) space by equations (3). First we state the relationship between the integrands of the integrals in the two spaces. By differentiating this relation n components of a covariant tensor are obtained. With these results it is easy to find the covariant tensor of rank 1 which is associated with Euler's equations.

In the (u, v) space let the integral to be minimized be

$$I = \int_{u_1}^{u_2} f(u, v, v')\,du. \tag{8}$$

The integrals in (1) and (8) will have the same value provided they are taken over corresponding arcs in the two spaces and provided

$$f(u, v, v') = F(x, y, y')\,dx/du, \tag{9}$$

in which the right side is a function of (u, v, v') by means of equations (3) and (5).

Differentiation of relation (9) with respect to v_i' shows that

$$f_{v_i'} = (F - y_\alpha' F_{y_\alpha'})\partial x/\partial v_i + F_{y_\alpha'}\,\partial y_\alpha/\partial v_i \qquad (i = 1, \cdots, n), \tag{10}$$

and then it can be verified that

$$(f - v_\alpha' f_{v_\alpha'}) = (F - y_\alpha' F_{y_\alpha'})\partial x/\partial u + F_{y_\alpha'}\,\partial y_\alpha/\partial u. \tag{11}$$

Equations (10) and (11) and the definition of a covariant tensor of rank 1 establish the following theorem.

THEOREM 1. *The functions* $(F - y_\alpha' F_{y_\alpha'})$, $F_{y_i'}$ $(i = 1, \cdots, n)$ *are the components of a covariant tensor of rank* 1.

COROLLARY. *The function*

$$(F - y_\alpha' F_{y_\alpha'})\,dx + F_{y_\alpha'}\,dy_\alpha,$$

in terms of which the transversality condition is stated, is an invariant.

For the differentials dx, dy_i form a contravariant tensor of rank 1. The inner product † of this tensor and the tensor in Theorem 1 gives the expression in the transversality condition which is, therefore, an invariant.

* L. P. Eisenhart, *Riemannian Geometry*, p. 10.

† L. P. Eisenhart, *loc. cit.*, p. 13.

The tensor associated with the Euler differential equations * may now be obtained. The Euler equations for the integral I are $df_{v_i'}/du - f_{v_i} = 0$. The expressions in these equations can be computed with the help of formulas (9) and (10) and hence we find for $i = 1, \cdots, n$, after some computation,

$$(12) \qquad \frac{d}{du} f_{v_i'} - f_{v_i} = \left[- y_a' \left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) \frac{\partial x}{\partial v_i} + \left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) \frac{\partial y_a}{\partial v_i} \right] \frac{dx}{du} .$$

These equations together with the equation formed by multiplying the above equations by $(-v_i')$ $i = 1, \cdots, n$ respectively and adding give the following theorem.

THEOREM 2. *The* $n + 1$ *functions*

$$-\left(\frac{d}{dx} F_{y_a'} - F_{y_a} \right) dy_a, \left(\frac{d}{dx} F_{y_i'} - F_{y_i} \right) dx \qquad (i = 1, \cdots, n)$$

are the components of a covariant tensor of rank 1.

3. *Canonical variables, equations, and transformations.* In this section we consider canonical variables and the Hamilton function. The Euler differential equations are replaced by canonical equations. Lastly, a discussion of canonical transformations is given with an example in which the variables are canonical.

The canonical variables † (x, y, z) which are introduced by the equations

$$(13) \qquad z_i = F_{y_i'}(x, y, y') \qquad (i = 1, \cdots, n)$$

are used in place of the element (x, y, y'). If the functional determinant of these equations with respect to y_i' is different from zero, we expect to find solutions $y_i' = P_i(x, y, z)$. Now the definition of the Hamilton function is

$$(14) \qquad H(x, y, z) \equiv y_a' F_{y_a'} - F,$$

in which $P_i(x, y, z)$ is substituted for y_i'. We may prove readily that $P_i = H_{z_i}$ so that the solutions of equations (13) for y_i' in terms of (x, y, z) are also expressible in the form $y_i' = H_{z_i}$. In changing from the element (x, y, y') to the canonical variables (x, y, z) the following property is important. Every solution $y_i(x)$ of class ‡ C'' of the Euler differential equa-

* O. Bolza, *Vorlesungen über Variationsrechnung*, p. 51.

† Riemann-Weber, *loc. cit.*, p. 186.

‡ O. Bolza, *loc. cit.*, p. 13.

tions defines a set of functions $y_i(x)$, $z_i(x)$ by means of equations (13) which satisfy the canonical equations *

$$(15)\qquad dy_i/dx - H_{z_i} = 0,\quad dz_i/dx + H_{y_i} = 0 \qquad (i = 1, \cdots, n),$$

and conversely.

Let u, v_i, w_i and $h(u, v, w)$ represent the canonical variables and the Hamilton function in the transformed space. After inserting in equations (10) and (11) the notations of canonical variables we can solve for H and z_i. This gives the equations †

$$(16)\qquad \begin{aligned} -H &= -h\partial u/\partial x + w_a \partial v_a/\partial x, \\ z_i &= -h\partial u/\partial y_i + w_a \partial v_a/\partial y_i \qquad (i = 1, \cdots, n). \end{aligned}$$

We may state now Theorem 1 in terms of canonical variables. It says that the set of $n + 1$ functions, $-H$, z_i $(i = 1, \cdots, n)$, are the components of a covariant tensor of rank 1.

If canonical variables are used, the invariance of the expression in the transversality condition which is given in the corollary to Theorem 1 may be stated as follows:

$$(17)\qquad -hdu + w_a dv_a = -Hdx + z_a dy_a.$$

It will be shown that the transformation between the two $(2n + 1)$-dimensional spaces, (x, y, z) and (u, v, w), which is given by equations (3) and the last n equations in (16), has the property that the solutions of the canonical equations (15) of the original problem go into the solutions of the canonical equations of the transformed problem, a result which will be seen to depend upon formula (17). The transformation is then a somewhat special case of transformations which are called canonical.‡

The following is the definition of a canonical transformation.§ A transformation from (u, v, w) to (x, y, z) coördinates

$$(18)\qquad x = x(u, v, w),\quad y_i = y_i(u, v, w),\quad z_i = z_i(u, v, w) \qquad (i = 1, \cdots, n),$$

is said to be canonical if it is a one-to-one transformation with its functional determinant $\partial(x, y, z)/\partial(u, v, w)$ not zero, and if three functions $H(x, y, z)$, $h(u, v, w)$, $\Psi(x, y, z)$, exist for which the relation

* Riemann-Weber, *loc. cit.*, p. 191.

† Riemann-Weber, *loc. cit.*, p. 200.

‡ *Encyklopädie der Mathematischen Wissenschaften*, Band 2, Teil 1, Heft 1 (1915), pp. 343-346.

§ See, for example, Carathéodory's definition in Riemann-Weber, *loc. cit.*, pp. 201-204.

$$-hdu + w_a dv_a = -Hdx + z_a dy_a + d\Psi, \tag{19}$$

is identically satisfied.

We shall consider the problem of the calculus of variations which is connected with the integral

$$J = \int_{x_1}^{x_2} (-H + z_a y_a' + d\Psi/dx)\,dx.$$

Since the Euler equations of a function which is the derivative of another function are identically zero,* the function $d\Psi/dx$ may be omitted in calculating these equations for this problem. The Euler equations are found to be the canonical equations

$$dz_i/dx + H_{y_i} = 0, \quad -(dy_i/dx - H_{z_i}) = 0 \qquad (i = 1, \cdots, n).$$

Hence by Theorem 2 when the dependent variables are $y_1, \cdots, y_n, z_1, \cdots, z_n$ we have the theorem:

THEOREM 3. *The expressions*

$$-(H_{y_a} dy_a + H_{z_a} dz_a), \quad (dz_i/dx + H_{y_i})\,dx,$$
$$-(dy_i/dx - H_{z_i})\,dx \qquad (i = 1, \cdots, n),$$

are the $2n + 1$ *components of a tensor of rank* 1 *with respect to transformations from the* (x, y, z)*- to the* (u, v, w)*-space.*

The transformation equations for this tensor are deduced by Carathéodory in the reference cited in a quite different and much less simple manner.

A fundamental property of tensors is that if the components of a tensor are zero in one coördinate system, then they are zero in every other coördinate system. Thus we have the following corollary:

COROLLARY. *By means of a canonical transformation the solutions of the canonical equations* (15) *of the original problem are transformed into the solutions of the canonical equations of the new problem.*

Since the relation (17) is a special case of (19), the transformation defined by equations (3) and the last n equations in (16) comes under the definition of canonical transformations, provided that the functional determinant is not zero. Since the equations $h_{w_i} = v_i'$ are always satisfied, as indicated in the second paragraph of Section 3, and with the help of the value of the determinant

$$|\,\partial v_i/\partial y_j - v_i'\partial u/\partial y_j\,| = \mathfrak{J}^{-1} dx/du \quad (i, j = 1, \cdots, n), \tag{20}$$

* O. Bolza, *loc. cit.*, p. 153.

where J is the Jacobian (4), it can be shown from equations (3) and the last n equations (16) that the functional determinant of x, y_i, z_i with respect to u, v_i, w_i has the value

$$\partial(x, y, z)/\partial(u, v, w) = \mathfrak{J}\mathfrak{J}^{-1} dx/du = dx/du,$$

which is different from zero along the arcs considered.

4. *Tensor properties of the conditions of Weierstrass and Legendre.* The Weierstrass E-function, in terms of which the necessary condition of Weierstrass * is expressed, may be written in the form

$$E(x, y, y', Y') = F(x, y, Y') - Y_a' F_{y_a'}(x, y, y') + [y_a' F_{y_a'}(x, y, y') - F(x, y, y')],$$

in which Y' is the symbol for the set $(Y_1', \cdots, Y_n')$. In the (u, v) space we define

$$(21) \qquad e(u, v, v', V') = f(u, v, V') - V_a' f_{v_a'}(u, v, v') + [y_a' f_{v_a'}(u, v, v') - f(u, v, v')],$$

where V' is the symbol for the set $(V_1', \cdots, V_n')$.

Transform the element (u, v, v') by the transformation (6), while the equations of transformation for the sets V' and Y' are those for the sets v' and y' with v' and y' replaced by V' and Y' respectively. From relation (9) and the equations of transformation for the sets V' and Y' it is seen that

$$f(u, v, V') = \{F(x, y, Y')\}\{1/[\partial u/\partial x + (\partial u/\partial y_a) Y_a']\}.$$

If this result is substituted in formula (21) together with those given in formulas (10) and (11) and the equations of transformation for the set V', then we obtain an expression for e which can be reduced to the following form:

$$(22) \qquad e(u, v, v', V') = \{E(x, y, y', Y')\}\{\partial x/\partial u + (\partial x/\partial v_a) V_a'\}.$$

This proves the theorem:

THEOREM 4. *The Weierstrass E-function is transformed by every transformation* (3) *and associated equations so that equation* (22) *is identically satisfied.*

The statement of the necessary condition of Legendre † involves the quadratic form $F_{y_a' y_\beta'} \eta_a \eta_\beta$. Concerning this form we find

* Riemann-Weber, *loc. cit.*, p. 182.

† Riemann-Weber, *loc. cit.*, p. 183.

THEOREM 5. *The function*

$$F_{y_\alpha' y_\beta'} \eta_\alpha \eta_\beta (1/dx)$$

is an invariant if the variations η_i $(i=1,\cdots,n)$ *are transformed by the relations* (31) *given below.*

The proof is as follows. Consider the one-parameter family of arcs through points 1 and 2

$$y_i = y_i(x,a) \quad (x_1 \leqq x \leqq x_2;\ i=1,\cdots,n), \tag{23}$$

which contains the extremal E_{12} for $a=0$. Then the set of variations η_i along E_{12} are defined by the equations

$$\eta_i(x) = y_{ia}(x,0) \qquad (i=1,\cdots,n). \tag{24}$$

In the (u,v) space the equations of the family of curves (23) have to be found. Substitute equations (23) into the first equation of transformation (6) and obtain

$$u = u(x, y(x,a)). \tag{25}$$

If $du/dx \neq 0$, the solution for x will be given by $x = x(u,a)$. When this result and equations (23) are put in the first group of n equations in (6), we secure the equations

$$v_i = v_i[x(u,a),\ y(x(u,a),a)] \equiv v_i(u,a) \quad (i=1,\cdots,n), \tag{26}$$

which represent the curves (23) in the (u,v) space. And as before we define

$$\zeta_i(u) = v_{ia}(u,0) \qquad (i=1,\cdots,n). \tag{27}$$

In order to find the law of transformation between η_i and ζ_i differentiate equations (26) with regard to the parameter and then set $a=0$ so that

$$\zeta_i = [(dv_i/dx)x_a + (\partial v_i/\partial y_\alpha) y_{\alpha a}]_{a=0}. \tag{28}$$

If the solution for x of equation (25) is substituted in (25), an identity is procured which is differentiated with respect to the parameter. This gives the relation

$$0 = [(du/dx)x_a + (\partial u/\partial y_\alpha) y_{\alpha a}]_{a=0}. \tag{29}$$

When the value of x_a from (29) and the definitions of η_i are put in (28), we find

$$\zeta_i = (\partial v_i/\partial y_\alpha - v_i' \partial u/\partial y_\alpha)\eta_\alpha \qquad (i=1,\cdots,n). \tag{30}$$

The determinant of the coefficients of η_i is the determinant in (20). By solving for η_i we have the desired law of transformation between η_i and ζ_i:

$$\eta_i = (\partial y_i/\partial v_a - y_i'\partial x/\partial v_a)\zeta_a \qquad (i = 1, \cdots, n). \tag{31}$$

By differentiating equations (10) with respect to v_j' it is found that

$$f_{v_i' v_j'} = F_{y_a' y_\beta'}(\partial y_a/\partial v_i - y_a'\partial x/\partial v_i)(\partial y_\beta/\partial v_j - y_\beta'\partial x/\partial v_j)\,du/dx.$$

Multiply by $\zeta_i\zeta_j$ and sum for $i, j = 1, \cdots, n$. On account of relations (31) this gives

$$f_{v_a' v_\beta'}\zeta_a\zeta_\beta(1/du) = F_{y_a' y_\beta'}\eta_a\eta_\beta(1/dx),$$

so that the theorem is proved.

5. *A tensor associated with Jacobi's differential equations.* A tensor associated with Jacobi's differential equations is deduced by a direct method although it can be found from the law of transformation of the integrand of the second variation of the integral J.

It is necessary to make a few preliminary remarks in regard to notations in the two spaces for Jacobi's equations and to derive some properties of the variations. If the family of arcs (23) is substituted in the integral J, the function $J(a)$ has the second derivative

$$J''(0) = \int_{x_1}^{x_2} (F_{y_a} y_{aaa} + F_{y_a'} y'_{aaa} + 2\Omega(x, \eta, \eta'))\,dx,$$

where the function 2Ω is defined as follows:

$$2\Omega = F_{y_a y_\beta}\eta_a\eta_\beta + 2F_{y_a y_\beta'}\eta_a\eta_\beta' + F_{y_a' y_\beta'}\eta_a'\eta_\beta'\,. \tag{32}$$

Since the family of arcs (23) contains the extremal E_{12} for $a = 0$, the Euler equations $dF_{y_i'}/dx - F_{y_i} = 0$ are satisfied for $a = 0$. This result and the vanishing of the values of $y_{iaa}(x, 0)$ at the limits x_1 and x_2 (see equations (37)) give us finally, by integrating the second term of the integrand in $J''(0)$ by parts,

$$J''(0) = \int_{x_1}^{x_2} 2\Omega(x, \eta, \eta')\,dx. \tag{33}$$

The Jacobi differential equations * are by definition the equations

$$d\Omega_{\eta_i'}/dx - \Omega_{\eta_i} = 0 \qquad (i = 1, \cdots, n). \tag{34}$$

If in the (u, v) space the corresponding family of arcs (26) is substituted in the integral I, the function $I(a)$ is obtained. By a reduction

* G. A. Bliss, *Calculus of Variations*, First Carus Monograph, p. 163.

which is similar to that for the second variation in the (x, y) space we find that

$$I''(0) = \int_{u_1}^{u_2} 2\omega(u, \zeta, \zeta')\,du.$$

where 2ω has the form

$$2\omega = f_{v_\alpha v_\beta}\zeta_\alpha\zeta_\beta + 2f_{v_\alpha v_\beta'}\zeta_\alpha\zeta_\beta' + f_{v_\alpha' v_\beta'}\zeta_\alpha'\zeta_\beta'. \tag{35}$$

The Jacobi differential equations in this space are

$$d\omega_{\zeta_i'}/du - \omega_{\zeta_i} = 0 \qquad (i = 1, \cdots, n). \tag{36}$$

Let the sets y_{i1}, y_{i2} denote the values of y_i $(i = 1, \cdots, n)$ for x_1 and x_2 respectively. Since the curves whose equations are given in (23) go through the points 1 and 2, we have $y_{i1} = y_i(x_1, a)$, $y_{i2} = y_i(x_2, a)$. By differentiation with respect to a we find

$$\begin{aligned} 0 &= y_{ia}(x_1, 0), & 0 &= y_{ia}(x_2, 0), \\ 0 &= y_{iaa}(x_1, 0), & 0 &= y_{iaa}(x_2, 0) \qquad (i = 1, \cdots, n). \end{aligned} \tag{37}$$

The set of variations (24) is now seen to satisfy the conditions

$$\eta_i(x_1) = \eta_i(x_2) = 0 \qquad (i = 1, \cdots, n). \tag{38}$$

Since the values u_1 and u_2 of the independent variable u in the (u, v) space locate points in that space which correspond respectively to the points determined by x_1 and x_2 in the (x, y) space, then the conditions (38) should imply $\zeta_i(u_1) = \zeta_i(u_2) = 0$, and conversely. This result is proved if it is observed that the determinant of the coefficients of η_i in (30) has the value $\mathcal{J}^{-1}dx/du$ which is not zero. The converse is proved similarly by means of equations (31).

In deriving the tensor associated with Jacobi's equations (34) equations (12) are used. In these equations substitute for y_i and v_i the equations of the corresponding families of arcs (23) and (26) respectively. After this result is differentiated with regard to a, let a be set equal to zero. Since the arc E_{12} obtained for $a = 0$ is an extremal, we note that the Euler equations $dF_{y_i'}/dx - F_{y_i} = 0$ are satisfied for $a = 0$. By means of this fact and the definitions of the functions Ω and ω it is easily shown that

$$\begin{aligned} \frac{d}{du}\,\omega_{\zeta_i'} - \omega_{\zeta_i} &= -y_\alpha'\left(\frac{d}{dx}\,\Omega_{\eta_\alpha'} - \Omega_{\eta_\alpha}\right)\frac{dx}{du}\,\frac{\partial x}{\partial v_i} \\ &\quad + \left(\frac{d}{dx}\,\Omega_{\eta_\alpha'} - \Omega_{\eta_\alpha}\right)\frac{dx}{du}\,\frac{\partial y_\alpha}{\partial v_i} \qquad (i = 1, \cdots, n). \end{aligned} \tag{39}$$

These equations together with the tensor equation which is secured by multi-

plying equations (39) by $-v_i'$ $(i=1, \cdots, n)$, respectively, and adding establish the following theorem:

THEOREM 6. *The $n+1$ functions*

$$-y_a'(d\Omega_{\eta_a r'}/dx - \Omega_{\eta_a})dx, \quad (d\Omega_{\eta_i'}/dx - \Omega_{\eta_i})dx \qquad (i=1, \cdots, n),$$

are the components of a covariant tensor of rank 1.

6. *Tensor properties of the determinants which determine conjugate points.* Conjugate points in Jacobi's condition can be determined easily in either one of two ways, involving in the one case the complete $2n$-parameter family of extremals and in the other case an n-parameter family of extremals which pass through the fixed point 1 and contain the extremal E_{12}. In each case the conjugate points are defined by the zeros of a determinant.* We will find the law of transformation of each determinant when the variables are subjected to a transformation (3).

In the first place we will consider the complete $2n$-parameter family of extremals

$$(40) \qquad y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n) \equiv y_i(x, a, b) \quad (i=1, \cdots, n),$$

which contains the extremal E_{12} for the set of parameter values $(a_0, b_0) \equiv (a_{10}, \cdots, a_{n0}, b_{10}, \cdots, b_{n0})$. The points 3 conjugate to point 1 on the extremal E_{12} are determined by the zeros $x \neq x_1$ of the determinant

$$(41) \qquad D(x, x_1, a_0, b_0) = \begin{vmatrix} y_{ia_j} & y_{ib_j} \\ y_{ia_j}(1) & y_{ib_j}(1) \end{vmatrix} \quad (i, j = 1, \cdots, n).$$

In this determinant and following expressions where the parameters are not indicated, they are assumed to have the values given by the set (a_0, b_0). Also when the value of a function is taken at point 1 we write $y_{ia_j}(1)$ for example.

In changing to (u, v) coördinates it is necessary to find the equations of the family of extremals in the (u, v) space which correspond to the family (40). Using the method by which the equations (26) were obtained the equations of the family of curves (40) in the (u, v) space are found to be

$$(42) \qquad v_i = v_i[x(u, a, b), y(x, a, b)] \equiv v_i(u, a, b) \qquad (i=1, \cdots, n).$$

The conjugate points on the corresponding extremal e_{12} are determined by the zeros $u \neq u_1$ of the determinant

* G. A. Bliss, *loc. cit.*, pp. 148-151.

$$
(43) \qquad d(u, u_1, a_0, b_0) = \begin{vmatrix} v_{ia_j} & v_{ib_j} \\ v_{ia_j}(1) & v_{ib_j}(1) \end{vmatrix} \qquad (i, j = 1, \cdots, n).
$$

In order to transform determinant (41) the following results will be needed. As equations (29) and (28) were found in Section 4, we find now the equations (44) when the set of parameters (a, b) is given the value (a_0, b_0) of the extremal E_{12}:

$$
(44) \qquad \begin{aligned}
0 &= (du/dx)x_{a_j} + (\partial u/\partial y_a)y_{aa_j}, \\
0 &= (du/dx)x_{b_j} + (\partial u/\partial y_a)y_{ab_j}, \qquad && (j = 1, \cdots, n), \\
v_{ia_j} &= (dv_i/dx)x_{a_j} + (\partial v_i/\partial y_a)y_{aa_j}, \\
v_{ib_j} &= (dv_i/dx)x_{b_j} + (\partial v_i/\partial y_a)y_{ab_j}, && (i, j = 1, \cdots, n).
\end{aligned}
$$

Since each of the expressions for $v_{ia_j}(u, a_0, b_0)$ and $v_{ib_j}(u, a_0, b_0)$ contains $n + 1$ terms, it is advisable to increase the order of the determinant (43) from $2n$ to $2n + 2$. With the help of equations (44) this determinant may be written as follows:

$$
d(u, u_1, a_0, b_0) =
$$

$$
\begin{vmatrix}
1 & \dfrac{du}{dx}x_{a_j} + \dfrac{\partial u}{\partial y_a} y_{aa_j} & 0 & \dfrac{du}{dx}x_{b_j} + \dfrac{\partial u}{\partial y_a} y_{ab_j} \\
\dfrac{dv_i}{du} & \dfrac{dv_i}{dx}x_{a_j} + \dfrac{\partial v_i}{\partial y_a} y_{aa_j} & 0 & \dfrac{dv_i}{dx}x_{b_j} + \dfrac{\partial v_i}{\partial y_a} y_{ab_j} \\
0 & \left(\dfrac{du}{dx}\right)_1 x_{a_j}(1) + \left(\dfrac{\partial u}{\partial y_a}\right)_1 y_{aa_j}(1) & 1 & \left(\dfrac{du}{dx}\right)_1 x_{b_j}(1) + \left(\dfrac{\partial u}{\partial y_a}\right)_1 y_{ab_j}(1) \\
0 & \left(\dfrac{dv_i}{dx}\right)_1 x_{a_j}(1) + \left(\dfrac{\partial v_i}{\partial y_a}\right)_1 y_{aa_j}(1) & \left(\dfrac{dv_i}{du}\right)_1 & \left(\dfrac{dv_i}{dx}\right)_1 x_{b_j}(1) + \left(\dfrac{\partial v_i}{\partial y_a}\right)_1 y_{ab_j}(1)
\end{vmatrix}
$$

where the subscript 1 indicates that the value is taken at point 1. The determinant may now be factored easily so that the following relation is found:

$$
(45) \qquad d(u, u_1, a_0, b_0) = \mathfrak{J}^{-1}(\mathfrak{J}^{-1})_1 D(x, x_1, a_0, b_0) \frac{dx}{du} \left(\frac{dx}{du}\right)_1,
$$

where $\mathfrak{J}^{-1}$ is the Jacobian of the transformation (6) and the subscript 1 denotes that the value is taken at point 1.

We can state now the following theorem:

THEOREM 7. *When there is given a $2n$-parameter family of extremals which includes the extremal E_{12} for the set of parameters (a_0, b_0), the determinant $D(x, x_1, a_0, b_0)$ for the determination of points conjugate to point* 1 *on the extremal E_{12} is transformed as in formula* (45).

Secondly, we consider an n-parameter family of extremals $y_i = y_i(x, a_1, \cdots, a_n) \equiv y_i(x, a)$, which includes the extremal E_{12} for the particular para-

metric values $(a_0) \equiv (a_{10}, \cdots, a_{n0})$ and all the extremals pass through the point 1. The points 3 conjugate to point 1 on the extremal arc E_{12} are determined by the zeros $x \neq x_1$ of the determinant

$$\Delta(x, a_0) \equiv | y_{ia_j}(x, a_0) | \qquad (i, j = 1, \cdots, n).$$

In the (u, v) space the conjugate points on the extremal e_{12}, corresponding to the extremal E_{12}, are determined by the zeros $u \neq u_1$ of the determinant

$$\delta(u, a_0) \equiv | v_{ia_j}(u, a_0) | \qquad (i, j = 1, \cdots, n),$$

where $v_i = v_i(u, a)$ are the equations of the corresponding family of extremals. The method of the first case now shows the law of transformation of the determinant $\Delta(x, a_0)$ to be

$$\delta(u, a_0) = \mathfrak{J}^{-1} \Delta(x, a_0) dx/du. \tag{46}$$

Hence we have

THEOREM 8. *When there is given an n-parameter family of extremals through point* 1 *which includes the extremal E_{12} for the set of parameters* (a_0), *the determinant* $\Delta(x, a_0)$ *for the determination of points conjugate to point* 1 *on the extremal E_{12} is transformed as in formula* (46).

CAYLEY'S DEFINITION OF NON-EUCLIDEAN GEOMETRY.

By James Pierpont.

1. Cayley in his Sixth Memoir upon Quantics (1859) laid the foundations of non-euclidean geometry. Although this paper is often referred to, its true significance seems to have been entirely overlooked. Led by Klein, whose autographed lectures on Nicht Euklidische Geometrie (1892) have enjoyed the widest popularity, another point of view has been adopted. This however depends on finding a *projective* definition of coördinates and cross ratios; and this can be done rigorously only by an intricate piece of reasoning.

The purpose of the present paper is to develop the implications of Cayley's paper and to show how naturally they lend themselves to a systematic and rigorous treatment of this subject.

2. Cayley takes four numbers x_1, x_2, x_3, x_4 and regards their ratios $x_1 : x_2 : x_3 : x_4$ as defining a point whose coördinates are the x's. In the notes to volume 2 which Cayley prepared for his "Collected Papers" he says, p. 605;—"As to my memoir, the point of view was that I regarded "coördinates" not as distances or ratios of distances but as an assumed fundamental notion, not requiring or admitting of explanation." It is well to remember in this connection that the notion of an abstract group is also due to Cayley (1854).

A straight is defined as the points $x_i = la_i + mb_i$, $(i = 1, 2, 3, 4)$, where a_i, b_i are the coördinates of two points a, b, and l, m are parameters. A plane is defined by $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$. It is easy to show that if a, b are two points of a plane, then all points on their join also lie on the plane. Also all points common to two planes lie on a straight.

To define distance and angle Cayley introduces the quadratic forms

$$(1)\qquad (x,x) = F(x,x) = \sum a_{ij}x_ix_j, \qquad a_{ij} = a_{ji}, \qquad (i,j = 1,2,3,4)$$

of non-vanishing determinant $a = \det|a_{ij}|$ and

$$(2)\qquad G(u,u) = \sum a^{ij}u_iu_j, \qquad (a^{ij} = \text{Minor of } a_{ij}/a).$$

The distance δ between two points x, y we define by

$$(3)\qquad \cos(\delta/c) = F(x,y)/[F(x,x)F(y,y)]^{1/2},$$

and the angle ϕ between two planes u, v we define by

$$(4)\qquad \cos(\phi/c') = G(u,v)/[G(u,u)G(v,v)]^{1/2},$$

Cayley takes $c = c' = 1$. We, following Klein, choose them so that δ, ϕ may be real when possible. From (3) we have

$$(5) \qquad \sin(\delta/c) = \Delta^{1/2}/[F(x,x)F(y,y)]^{1/2}, \qquad (\Delta = F(x,x)F(y,y) - F(x,y)^2.$$

3. Let a, b, c be three points on a straight, let $\gamma = \mathrm{dist}(ab)$, $\eta = \mathrm{dist}(bc)$, $\delta = \mathrm{dist}(ac)$. We wish to show that $\delta = \gamma + \eta$. A similar proof holds for the addition of angles. Consider

$$\Delta = \cos[(\gamma + \eta)/c] = \cos(\gamma/c)\cos(\eta/c) - \sin(\gamma/c)\sin(\eta/c)$$

and let us write for the sake of brevity (a, b) as ab, etc. Then

$$\begin{aligned}\Delta = {} & [ab/(aa \cdot bb)^{1/2}]\,[bc/(bb \cdot cc)^{1/2}] \\ & -(aa \cdot bb - ab^2)^{1/2}(bb \cdot cc - bc^2)^{1/2}/(aa \cdot bb)^{1/2}(bb \cdot cc)^{1/2}.\end{aligned}$$

Let us denote the numerator of the last term on the right by $A^{1/2}$. Then

$$A = (aa \cdot bb - ab^2)(bb \cdot cc - bc^2).$$

Now

$$\begin{aligned} b &= \alpha a + \beta c, & (bb) &= \alpha^2(aa) + 2\alpha\beta(ac) + \beta^2(cc) \\ (ab) &= \alpha(aa) + \beta(ac), & (bc) &= \alpha(ac) + \beta(cc) \end{aligned}$$

$$aa \cdot bb - ab^2 = \beta^2(aa \cdot cc - ac^2)$$
$$bb \cdot cc - bc^2 = \alpha^2(aa \cdot cc - ac^2)$$
$$A = \alpha^2\beta^2(aa \cdot cc - ac^2)^2.$$

Hence

$$\Delta = [ab \cdot bc - \alpha\beta(aa \cdot cc - ac^2)]/bb(aa \cdot cc)^{1/2} = B/C,$$
$$ab \cdot bc = \alpha^2 \cdot aa \cdot ac + \alpha\beta(ac^2 + aa \cdot cc) + \beta^2\, ac \cdot cc.$$

Hence

$$B = \alpha^2 \cdot aa \cdot ac + 2\alpha\beta\, ac^2 + \beta^2\, ac \cdot cc = (ac)(bb)$$

Thus

$$\Delta = (ac)(bb)/(bb)[(aa)(cc)]^{1/2} = (ac)/[(aa)(cc)]^{1/2} = \cos(\delta/c).$$

Hence

$$\cos(\delta/c) = \cos[(\gamma + \eta)/c] \quad \text{or} \quad \delta = \gamma + \eta.$$

We can now express the parameters in $x_i = \alpha a_i + \beta b_i$. We find at once

$$(6) \qquad \begin{aligned} \alpha &= [(xx)/(aa)]^{1/2} \sin[(x,b)/c]/\sin[(a,b)/c], \\ \beta &= [(xx)/(bb)]^{1/2} \sin[(a,x)/c]/\sin[(a,b)/c], \end{aligned}$$

where we have set $(x, b) = \mathrm{dist}(x, b)$, etc.

In fact

$$(xx) = \alpha^2(aa) + 2\alpha\beta(ab) + \beta^2(bb).$$

Let $\gamma = \operatorname{dist}(a, x) \qquad \delta = \operatorname{dist}(a, b),$

then by (3)

$$(7) \qquad (xx) = \alpha^2(aa) + \beta^2(bb) + 2\alpha\beta[(aa)(bb)]^{1/2}\cos(\delta/c).$$

Also by (3)

$$[(aa)(xx)]^{1/2}\cos(\gamma/c) = (ax) = \alpha(aa) + \beta(ab)$$
$$= \alpha(aa) + \beta[(aa)(bb)]^{1/2}\cos(\delta/c).$$

Hence squaring

$$(xx)\cos^2(\gamma/c) = \alpha^2(aa) + 2\alpha\beta[(aa)(bb)]^{1/2}\cos(\delta/c) + \beta^2(bb)\cos^2(\delta/c),$$

or using (7)

$$(xx)\cos^2(\gamma/c) = (xx) - \beta^2(bb) + \beta^2(bb)\cos^2(\delta/c)$$

or

$$(xx)\sin^2(\gamma/c) = \beta^2(bb)\sin^2(\delta/c).$$

This is the second equation in (6); we get α similarly.

4. The plane

$$(8) \qquad (g, x) = \sum a_{ij}g_ix_j = \sum_j x_j \sum_i a_{ij}g_i = \sum x_ju_j = 0$$

we call the absolute polar of g, or simply the polar of g. The distance between g and any point x on this plane is such that $\cos\delta/c = 0$. Hence $\delta = \pi c/2$. We call g the pole of (8). We ask how many poles g, does a plane

$$(9) \qquad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0,$$

have i. e. how many points g are there such that the polar of g, $(gx) = 0$, is identical with (9). For (8) to be identical with (9) it is necessary and sufficient that

$$\rho c_j = a_{j1}g_1 + a_{j2}g_2 + a_{j3}g_3 + a_{j4}g_4.$$

Since the determinant a of $F(xx)$ is $\neq 0$, we have

$$(10) \qquad g_i = \rho(a^{i1}c_1 + a^{i2}c_2 + a^{i3}c_3 + a^{i4}c_4).$$

Thus $g_1 : g_2 : g_3 : g_4$ is uniquely determined. The plane (9) has thus a unique pole whose coördinates are given by (10).

The equations of the straight joining the points a, b are $x_i = \alpha a_i + \beta b_i$. Suppose b lies on the polar of a. Set distance $(a, x) = s$, then distance $(x, b) = \pi c/2 - s$. Then (6), (7) give

$$\alpha = [(xx)/(aa)]^{1/2}\cos(s/c), \quad \beta = [(xx)/(bb)]^{1/2}\sin(s/c),$$
$$\beta/\alpha = [(aa)/(bb)]^{1/2}\tan(s/c).$$

The point x on the straight lies on the plane $L(x) = u_1x_1 + \cdots u_4x_4 = 0$,

when α, β are such that $\alpha L(a) + \beta L(\beta) = 0$, which determines β/α unless $L(b) = 0$, in which case $x = b$. Thus a straight cuts a plane but once.

When b lies on the polar of a we say b is at a quadrant's distance from a.

5. The distance δ from a point x to the plane (9) we define in this way: Let g be the pole of (9). The join of g, x cuts (9) in a point h and by definition distance $(g, h) = \pi c/2$. We define $\delta = \text{dist}(x, h)$. Let $\eta = \text{dist}(g, x)$, then $\eta + \delta = \pi c/2$. Now

$$\cos(\eta/c) = (g, x)/[(g, g)(x, x)]^{\frac{1}{2}} = \cos[(\pi c/2 - \delta)/c] = \sin(\delta/c).$$

Thus

$$(11) \qquad \sin(\delta/c) = (g, x)/[(g, g)(x, x)]^{\frac{1}{2}}.$$

This expression involves $g_1 \cdots g_4$ which are not given directly. Now

$$(12) \qquad (g, x) = \sum a_{ij} g_i x_j = \sum_j x_j \sum_i a_i g_j = \sum_j x_j \sum_i a_{ij} \rho \sum_k a^{ik} c_k$$
$$= \rho \sum_j x_j \sum_k c_k \sum_i a_{ij} a^{ik} = \rho \sum_j c_j x_j.$$

$$(13) \qquad (g, g) = \sum a_{ij} g_i g_j = \rho^2 \sum_{ij} a_{ij} \sum_\alpha a^{i\alpha} c_\alpha \sum_\beta a^{j\beta} c_\beta$$
$$= \rho^2 \sum_{\alpha\beta} c_\alpha c_\beta \sum_i a^{i\alpha} \sum_j a_{ij} a^{j\beta} = \rho^2 \sum_{\alpha\beta} a^{\alpha\beta} c_\alpha c_\beta$$
$$= \rho^2 G(c, c).$$

These in (11) give

$$(14) \qquad \sin(\delta/c) = \rho(c_1 x_1 + \cdots + c_4 x_4)/[F(g, g)F(x, x)]^{\frac{1}{2}}$$
$$= (c_1 x_1 + \cdots + c_4 x_4)/[F(x, x)G(c, c)]^{\frac{1}{2}}.$$

The four planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ define a tetrahedron which we call τ. The distance δ_i of a point x to the plane x_i is given by

$$(15) \qquad \sin(\delta_i/c) = x_i/(a^{ii})^{\frac{1}{2}}[(x, x)]^{\frac{1}{2}}.$$

Thus

$$x_1 : x_2 : x_3 : x_4$$
$$= (a^{11})^{\frac{1}{2}} \sin(\delta_1/c) : (a^{22})^{\frac{1}{2}} \sin(\delta_2/c) : (a^{33})^{\frac{1}{2}} \sin(\delta_3/c) : (a^{44})^{\frac{1}{2}} \sin(\delta_4/c).$$

This gives a geometric interpretation of the coördinates of a point in terms of distance.

Let V_i be the vertex opposite the face $x_i = 0$ of τ; i. e. the point whose coördinates v are all $= 0$ except v_i. Its polar plane is $(v, x) = 0$

$$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} x_4 = 0.$$

The distance η_i of V_i to the plane

(16) $$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0$$

is given by

$$\sin(\eta_i/c) = u_i/(a_{ii})^{1/2}[G(u,u)]^{1/2}.$$

Hence

$$u_i = (a_{ii})^{1/2}[G(uu)]^{1/2}\sin(\eta_i/c).$$

Hence

(17) $$u_1 : u_2 : u_3 : u_4 \\ = (a_{11})^{1/2}\sin(\eta_1/c) : (a_{22})^{1/2}\sin(\eta_2/c) : (a_{33})^{1/2}\sin(\eta_3/c) : (a_{44})^{1/2}\sin(\eta_4/c).$$

6. The angle ϕ between the two planes

(18) $$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \quad v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = 0,$$

is given by (4). Let us call g, h the poles of these planes; Then

$$g_i = \rho \sum_\alpha a^{i\alpha}u_\alpha, \qquad h_j = \rho \sum_\beta a^{j\beta}v_\beta.$$

The distance δ between g, h is given by

$$\cos(\delta/c) = \sum_{ij} a_{ij}g_ih_j/[F(g,g)F(h,h)]^{1/2},$$

$$\sum_{ij} a_{ij}g_ih_j = \rho^2 \sum_{ij} a_{ij} \sum_\alpha a^{i\alpha}u_\alpha \sum_\beta a^{j\beta}v_\beta = \rho^2 \sum_{\alpha\beta} u_\alpha v_\beta \sum_i a^{i\alpha} \sum_j a_{ij}a^{j\beta}$$
$$= \rho^2 \sum_{\alpha\beta} a^{\alpha\beta}u_\alpha v_\beta = \rho^2 G(u,v).$$

Hence using (13)

$$\cos(\delta/c) = G(u,v)/[G(u,u)G(v,v)]^{1/2}.$$

Comparing this with (4) gives

(19) $$\cos(\phi/c') = \cos(\delta/c) \quad \text{or} \quad \phi = c'\delta/c.$$

When $\phi = \pi/2$ we say the planes (18) are orthogonal; the distance between their poles is

(20) $$\delta = (\pi/2)c/c'.$$

The angle ϕ_{ij} between the faces $x_i = 0$, $x_j = 0$ of the tetrahedron τ is given by

(21) $$\cos(\phi_{ij}/c') = a^{ij}/(a^{ii}a^{jj})^{1/2}.$$

We define the angle θ between two straights l, m meeting at a point a in this way. Let b, c be points on l, m at a quadrants distance from a. If β, γ are the polar planes of b, c we define θ as angle between β, γ. Now we saw the distance δ between b, c is related to θ by $\phi/c' = \delta/c$. Hence we can define θ as c/c' times the distance between b, c.

7. Let us introduce new variables ξ_1, ξ_2, ξ_3, ξ_4 setting

$$x_i = \sum_j c_{ij}\xi_j, \tag{22}$$

where $c = \det | c_{ij} | \neq 0$. Setting $c^{ij} =$ minor c_{ij}/c, we have

$$\xi_i = \sum_j c^{ji}x_j. \tag{23}$$

Then

$$\begin{aligned} F(x,x) &= \sum a_{ij}x_ix_j = \sum_{ij} a_{ij} \sum_\lambda c_{i\lambda}\xi_\lambda \sum_\mu c_{j\mu}\xi_\mu \\ &= \sum_{\lambda\mu} \xi_\lambda\xi_\mu \sum_{ij} a_{ij}c_{i\lambda}c_{j\mu}. \end{aligned}$$

Hence setting

$$\alpha_{\lambda\mu} = \sum_{ij} a_{ij}c_{i\lambda}c_{j\mu}, \tag{24}$$

$$F(x,x) = \sum_{\lambda\mu} \alpha_{\lambda\mu}\xi_\lambda\xi_\mu = \Phi(\xi,\xi). \tag{25}$$

The distance δ between two points g, h is given by

$$\cos(\delta/c) = \sum a_{ij}g_ih_j/[F(g,g)F(h,h)]^{1/2}. \tag{26}$$

Suppose (23) makes γ, η correspond to g, h, then conversely (22) gives

$$g_i = \sum_\lambda c_{i\lambda}\gamma_\lambda, \qquad h_j = \sum_\mu c_{j\mu}\eta_\mu.$$

These in (26) give

$$\cos(\delta/c) = [\sum_{ij} a_{ij} \sum_\lambda c_{i\lambda}\gamma_\lambda \sum_\mu c_{j\mu}\eta_\mu]/[\Phi(\gamma,\gamma)\Phi(\eta,\eta)]^{1/2} = N/D.$$

Now by (20)

$$N = \sum_{\lambda\mu} \gamma_\lambda\eta_\mu \sum_{ij} a_{ij}c_{i\lambda}c_{j\mu} = \sum_{\lambda\mu} \gamma_\lambda\eta_\mu\alpha_{\lambda\mu}.$$

Thus

$$\cos(\delta/c) = \Phi(\gamma,\eta)/[\Phi(\gamma,\gamma)\Phi(\eta,\eta)]^{1/2}, \tag{27}$$

i. e. $\cos\delta/c$ is an invariant of the quadratic form (1) relative to the linear transformations (22). The plane

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \tag{28}$$

becomes in the new variables

$$\sum v_i\xi_i = 0 \quad \text{where} \quad v_i = \sum_\lambda c_{\lambda i}u_\lambda. \tag{29}$$

The distance δ of point x to the plane (28) is given by

$$\sin(\delta/c) = \rho(u_1x_1 + \cdots + u_4x_4)/[F(g,g)F(x,x)]^{1/2},$$

where g is the pole of (28).

Passing to the ξ variables this gives

(30) $$\sin(\delta/c) = \rho(v_1\xi_1 + \cdots + v_4\xi_4)/[\Phi(\gamma,\gamma)\Phi(\xi,\xi)]^{1/2},$$

where γ corresponds to g. Thus $\sin(\delta/c)$ is an invariant of the form (1). Now

$$\Phi(\gamma,\gamma) = \sum_{ij} \alpha_{ij}\gamma_i\gamma_j = \rho^2 \sum_{ij} \alpha_{ij} \sum_\lambda \alpha^{i\lambda} v_\lambda \sum_\mu \alpha^{j\mu} v_\mu = \rho^2 \sum_{\lambda\mu} \alpha^{\lambda\mu} v_\lambda v_\mu.$$

Hence

(31) $$\sin(\delta/c) = (v_1\xi_1 + \cdots + v_4\xi_4)/[\Psi(v,v)\Phi(\xi,\xi)]^{1/2},$$

where

$$\Psi(v,v) = \sum_{\lambda\mu} \alpha^{\lambda\mu} v_\lambda v_\mu.$$

If δ_i is the distance from Point P $(\xi_1, \xi_2, \xi_3, \xi_4)$ to the plane $\xi_i = 0$, we have

(32) $$\sin(\delta_i/c) = \xi_i/[\alpha^{ii}\Phi(\xi,\xi)]^{1/2},$$

or

(33) $$\xi_1 : \xi_2 : \xi_3 : \xi_4 = (\alpha^{11})^{1/2}\sin(\delta_1/c) : (\alpha^{22})^{1/2}\sin(\delta_2/c) : (\alpha^{33})^{1/2}\sin(\delta_3/c) : (\alpha^{44})^{1/2}\sin(\delta_4/c).$$

Geometrically the transformation (22) or (23) represents replacing the tetrahedron of reference τ by a tetrahedron τ' whose faces are $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$.

Let us refer the straight

(34) $$x_i = la_i + mb_i, \qquad (i = 1, 2, 3, 4),$$

to the τ' tetrahedron. By (23), $\xi_i = \sum_j c^{ji} x_j$.

As x lies on (34),

$$\xi_i = \sum_j c^{ji}(la_i + mb_i)$$
$$= l\alpha_i + m\beta_i,$$

where by (23), α_i, β_i are the coördinates a, b referred to τ'. From algebra we know that a real transformation (23) will reduce the form (1) to one of three types

(35) $$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 0,$$

(36) $$\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 = 0,$$

(37) $$\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 = 0.$$

To (35) corresponds the geometry first discovered by Riemann. To (36)

corresponds the geometry discovered by Lobachevsky and Bolyai. Finally to (37) corresponds a geometry which I have called L-geometry.*

8. The distance $\delta = ds$ between x and $x + dx$ is given by

$$\cos(ds/c) = (x, x + dx)/[(x, x)(x + dx, x + dx)]^{1/2} = A/B = 1 - ds^2/2c^2 + \cdots.$$

Now

$$\begin{aligned}
(x, x + dx) &= (x, x) + (x, dx),\\
(x + dx, x + dx) &= (x, x) + 2(x, dx) + (dx, dx),
\end{aligned}$$

$$\begin{aligned}
B &= (x, x)[1 + 2(x, dx)/(x, x) + (dx, dx)/(x, x)]^{1/2}.\\
A/B &= [1 + (x, dx)/(x, x)]\,[1 + 2(x, dx)/(x, x) + (dx, dx)/(x, x)]^{-1/2}\\
&= [1 + (x, dx)/(x, x)] \cdot [1 - (1/2)\{2(x, dx)/(x, x) + (dx, dx)/(x, x)\}\\
&\qquad + (3/8)\{2(x, dx)/(x, x) + (dx, dx)^2/(x, x)\} + \cdots],\\
&= 1 - (x, dx)/(x, x) - (dx, dx)/2(x, x) + (3/2)(x, dx)^2/(x, x)^2 + \cdots,\\
&\quad + (x, dx)/(x, x) - (x, dx)^2/(x, x)^2 - \cdots,\\
&= 1 - (dx, dx)/2(x, x) + (1/2)(x, dx)^2/(x, x)^2 + \cdots\\
&= 1 - ds^2/2c^2 + \cdots.
\end{aligned}$$

Hence

$$ds^2/c^2 = (dx, dx)/(x, x) - (x, dx)^2/(x, x)^2. \tag{38}$$

Since only the ratios $x_1 : x_2 : x_3 : x_4$ have been used, if we like, we may set

$$(x, x) = \sum a_{ij} x_i x_j = h^2, \quad \text{a constant.} \tag{39}$$

Then

$$(x, dx) = 0 \quad \text{and} \quad ds^2 = c^2/h^2 (dx, dx). \tag{40}$$

If we choose c, h so that $c = h$, then

$$ds^2 = \sum a_{ij} dx_i dx_j. \tag{41}$$

Let us define the angle θ between two curves meeting at a point a, as the angle between their tangents, whose equations are, say

$$x_i = a_i \cos s/c + b_i \sin s/c, \quad x_i' = a_i \cos s'/c + \beta_i \sin s'/c.$$

We have at the point a for which $s = s' = 0$,

$$dx_i/ds = b_i/c, \qquad dx_i'/ds' = \beta_i/c.$$

If $\delta = \text{dist}(b, \beta)$ we set $\theta = \delta/c$ and have

$$\cos(\delta/c) = \sum a_{ij} b_i \beta_j / [(b, b)(\beta, \beta)]^{1/2}$$

or

$$\cos\theta = \sum a_{ij} (dx_i/ds)(dx_j'/ds') [F(b, b) F(\beta, \beta)]^{1/2}.$$

* *Monatshefte für Mathematik und Physik*, Vol. 35 (1928).

If we suppose $(b, b) = (\beta, \beta) = c^2$ we have

(42) $$\cos\theta = \sum a_{ij}(dx_i/ds)(dx_j'/ds').$$

Let us find the curves for which

$$\delta \int ds = 0,$$

where ds is given by (41).

Performing the variation we get

$$\int ds \sum_j \delta x_j \sum_i a_{ij}\, d^2x_i/ds^2 = 0.$$

From the relation (39) we get $\sum a_{ij}x_i\delta x_j = 0$, hence introducing a Lagrangean multiplier H we get

$$\int ds \sum_j \delta x_j \sum_i a_{ij}\{d^2x_i/ds^2 + Hx_i\} = 0,$$

or

(43) $$\sum_i a_{ij}(d^2x_i/ds^2 + Hx_i) = 0, \qquad (j = 1, 2, 3, 4).$$

Multiplying these four equations respectively by x_1, x_2, x_3, x_4, and adding, gives

$$\sum_{ij} a_{ji}x_j\, d^2x_i/ds^2 + H \sum_{ij} a_{ij}x_ix_j = 0,$$

or

(44) $$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + Hc^2 = 0.$$

From (39) we have differentiating twice

$$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + \sum_{ij} a_{ij}(dx_i/ds)(dx_j/ds) = 0,$$

or

$$\sum_{ij} a_{ij}x_j\, d^2x_i/ds^2 + 1 = 0.$$

This in (44) gives $H = 1/c^2$ which put in (43) gives the four linear equations

$$\sum a_{ji}(d^2x_i/ds^2 + x_i/c^2) = 0, \qquad (j = 1, 2, 3, 4).$$

The determinant of the system of equations is $a \neq 0$. Hence this system admits only the solution

$$d^2x_i/ds^2 + x_i/c^2 = 0, \qquad (i = 1, 2, 3, 4).$$

Hence integrating

$$\begin{aligned} x_i &= a_i \cos s/c + b_i \sin(s/c) \\ &= \alpha a_i + \beta b_i, \end{aligned}$$

which are the equations of a straight as we saw § 4.

9. We have now gone far enough to show how the geometry belonging to the quadratic form (1) with non-vanishing determinant may be developed. It remains only to put this abstract geometry in relation to the geometry of our physical space.

In two papers,* I have considered the optics of space of constant curvature taking as fundamental assumption that the path of a ray of light in such a space is given by Fermat's principle, $\delta \int n ds = 0$. I showed that in such a space, light behaved in the main as in euclidean space. For constant index of refraction the path of a ray is a straight according to Cayley's definition. In elliptic and hyperbolic spaces it has long been known that rigid bodies exist which can be freely moved about without distortion, and I have shown the same holds in L-space in the paper referred to in § 7. Thus we have the means, partly mechanical and partly optical, to construct straight edges, planes etc., Also we are in position to construct measuring bars. This being so we can define a tetrahedron and define our coördinates by (15) where the a's are determined by one of the forms (35), (36), or (37).

The indentification of our abstract space with our physical space is thus complete. Cayley appears to have had no interest in identifying his abstract geometry with our physical space, his "coördinates" have thus left a fundamental notion undefined.

In Klein's theory straight lines are the undefinabled. They may be identified with what we call straights in our physical world in a manner similar to that just outlined.

Disregarding such identification, but looking at these geometries as abstractions the only question would then be;—Which of these two abstract methods of procedure Cayley's or Klein's is the more direct and easy. For those who are familiar with projective geometry when developed *projectively* † there would be no preference.

But for those who have not gone thru such a strenuous course of training the advantage, so it seems to the present writer, lies with Cayley.

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* *Transactions of the American Mathematical Society*, Vol. 30 (1927), pp. 33-48; *American Journal of Mathematics*, Vol. 49 (1927), pp. 343-354.

† As for example, in the magistral treatise of Veblen and Young.

FAMILIES OF PLANE INVOLUTIONS OF GENUS 2 OR 3.

By Franklin G. Williams.

1. *Introduction.* A general method for finding families of plane involutions of order t for a pencil of curves of genus $p \leq (t-1)(t-2)/2$ has been developed recently * and applied to the case $t=3$ so that $p=0$ or 1, and to the cases $t=4$, $p=0$ or 1. In this paper, the case of pencils with $t=4$ has been completed by considering pencils of genus 2 or 3. The irreducible types of pencils of curves with $p=2$ are known.† For $p=3$, they are determined in this paper.

2. *General Method.* In a plane x, let $|C_1|$ denote a linear system of curves of order n_1 with an $i_1, i_1' \cdots$ fold point at $P_i, P_i', \cdots$, and let $|C_2|$ denote a similar linear system. Then $|C_1| + |C_2|$ is defined as a linear system of order $n_1 + n_2$, having an $i_1 + i_1'$-fold point at P_i, P_i'. Any curve $|C_1|$ together with any curve $|C_2|$ constitute a composite curve of the system $|C_1| + |C_2|$. Given a pencil of curves

$$|C| = a_1C_1 + a_2C_2 = 0, \tag{1}$$

then $|C| + |\bar{C}|$ is a system containing $(a_1C_1 + a_2C_2)\bar{C} = 0$ as a composite pencil. Let

$$|C| + |\bar{C}| = (a_1C_1 + a_2C_2)\bar{C} + a_3C_4 + \cdots + a_{r+1}C_{r+1} = 0. \tag{2}$$

The symbol $[C_1, C_2]$ is used to denote the number of variable intersections of a curve of $|C_1|$ with a curve of $|C_2|$. Then $[C_1, C_1 + C_2] = [C_1, C_1] + [C_1, C_2]$.

We wish to consider the cases in which any curve of $|C|$ meets any curve of $|C| + |\bar{C}|$ in a certain number, t, of variable points. We are going to consider the possible cases with $t=4$ and $p=2$ or 3.

The irreducible pencils of genus 2, mentioned in Section 1, are:

$$C_4 : A^2 12B, \quad C_6 : 8A^2 4B, \quad C_7 : A^3 10B, \quad C_9 : 8A^3 2B^2 C, \quad C_{13} : A^5 9B^4,$$

where the subscript denotes the order of the curve C, and a superscript denotes the multiplicity of a corresponding point.

As typical of the process employed, let (1) be the pencil, $C_4 : A^2 12B$,

* F. R. Sharpe, *American Journal of Mathematics*, Vol. 50 (October, 1928).

† Michele di Franchis, *Rendiconti del Circolo Matematico di Palermo*, Vol. 13 (1899).

and $\bar{C}$ a line through A. Then $|C| + \bar{C}$ is the system, $C_5: A^3 10B$, for which $r = 4$.

Consider the pencil

$$z_2 C_1 - z_1 C_2 = 0, \tag{3}$$

and the algebraic system of $C_5: A^3 10B$

$$(b_1 z_3 + c_1) C_1 \bar{C} + (b_2 z_3 + c_2) C_2 \bar{C} + (b_3 z_3 + c_3) C_3 + (b_4 z_3 + c_4) C_4 + (b_5 z_3 + c_5) C_5 = 0 \tag{4}$$

where the b's are of arbitrary degrees $n-1$, n respectively in z_1, z_2. Given a point in (x), then (3) and (4) determine a point in (z); and, conversely, given a point in (z), then (3) and (4) determine a group of 4 points in (x). Thus (3) and (4) determine an involution of order 4, I_4.

By solving (3) and (4) for the ratio $z_1 : z_2 : z_3$, we have

$$z_1 = C_1 u; \quad z_2 = C_2 u; \quad z_3 = v \tag{5}$$

where u and v are linear in $C_1\bar{C}$, $C_2\bar{C}$, $C_3\bar{C}$, $C_4\bar{C}$, $C_5\bar{C}$ with coefficients of orders $n-1$, n respectively in C_1, C_2. Hence the lines of (z) correspond to the net of curves

$$a_1 C_1 u + a_2 C_2 u + C_3 v = 0. \tag{6}$$

The curves of the net (6) are therefore $C_{4n+5}: A^{2n+3} 10 B^{n+1} 2 C^n$, while $u = 0$ represents a curve $C_{4n+1}: A^{2n+1} 10 B^n 2 C^{n-1}$. Hence the curves $u = 0$ and $v = 0$ have

$$(4n+1)(4n+5) - (2n+3)(2n+1) - 10n(n+1) - 2n(n-1) = 8n + 2$$

intersections at points, D, other than at the basis points of $C_4: A^2 10 B 2 C$. We have therefore determined a family of involutions of the type

$$C_{4n+1}: A^{2n+1} 10 B^n 2 C^{n-1} (8n-6) D.$$

Following the method outlined above and making use of the tables of linear systems of curves given by Jung,* the following families of involutions of genus 2 have been found:

* G. Jung, *Annali di Matematico*, Ser. 2, Vols. 14, 15.

Pencil $C_4: A^2 12B$

$\bar{C}$	Genus	Families of Involutions
$(C_4: A\,10B)$	—	$C_{4n}: A^{2n-1}10B^n 2C^{n-1}(8n-7)D$
$(C_4: A^2 8B)$	—	$C_{4n}: A^{2n}8B^n 4C^{n-1}(8n-8)D$
C_1	—	$C_{4n+1}: A^{2n}12B^n(8n-3)C$
$C_1: A$	—	$C_{4n+1}: A^{2n+1}10B^n 2C^{n-1}(8n-6)D$
$C_2: 4B$	—	$C_{4n+2}: A^{2n}4B^{n+1}8C^n(8n-4)D$
$C_3: A6B$	1	$C_{4n+3}: A^{2n+1}6B^{n+1}6C^n(8n-6)D$
$C_3: 8B$	1	$C_{4n+3}: A^{2n}8B^{n+1}4C^n(8n-3)D$
$(C_3: A8B)^2$	1	$C_{4n+2}: A^{2n}8B^{n+1}4C^{n-1}(8n-12)D$
$C_4: AB^2 8C$	2	$C_{4n}: A^{2n-1}B^{n+1}8C^n 3D^{n-1}(8n-9)E$
$C_6: A^2 7B^2 2C$	2	$C_{4n+2}: A^{2n}7B^{n+1}2C^n 3D^{n-1}(8n-10)E$
$C_4: 12B$	3	$C_{4n}: A^{2n-2}12B^n(8n-8)C$
$C_5: AB^3 11C$	3	$C_{4n+1}: A^{2n-1}B^{n+2}11C^n(8n-8)D$
$C_6: A^2 6B^2 4C$	3	$C_{4n+2}: A^{2n}6B^{n+1}4C^n 2D^{n-1}(8n-8)E$
$C_7: A^3 9B^2$	3	$C_{4n+3}: A^{2n+1}9B^{n+1}3C^{n-1}(8n-8)D$
$C_7: A^2 B^3 8C^2 D$	3	$C_{4n+3}: A^{2n}B^{n+2}8C^{n+1}D^n 2E^{n-1}(8n-9)F$
$C_5: A^2 B^2 10C$	4	$C_{4n+1}: A^{2n}B^{n+1}10C^n D^{n-1}(8n-5)E$
$C_5: A2B^2 10C$	4	$C_{4n+1}: A^{2n-1}2B^{n+1}10C(8n-6)D$
$C_6: A^4 12B$	4	$C_{4n+2}: A^{2n+2}12B^n(8n-4)C$
$C_6: A^2 5B^2 6C$	4	$C_{4n+2}: A^{2n}5B^{n+1}6C^n D^{n-1}(8n-6)E$
$C_6: A6B^2 6C$	4	$C_{4n+2}: A^{2n-1}6B^{n+1}6C^n(8n-7)D$
$C_7: A^2 10B^2$	4	$C_{4n+3}: A^{2n}10B^{n+1}2C^{n-1}(8n-7)D$
$C_7: A11B^2$	4	$C_{4n+3}: A^{2n-1}11B^{n+1}C^{n-1}(8n-4)D$
$C_7: A^2 B^3 7C^2 3D$	4	$C_{4n+3}: A^{2n}B^{n+2}7C^{n+1}3D^n E^{n-1}(8n-7)F$
$C_7: AB^3 8C^2 3D$	4	$C_{4n+3}: A^{2n-1}B^{n+2}8C^{n+1}3D^n(8n-8)E$
$C_8: A^2 B^4 10C^2$	4	$C_{4n}: A^{2n-2}B^{n+2}10C^n D^{n-2}(8n-16)E$
$C_{10}: A^4 8B^3 2C^2$	4	$C_{4n+2}: A^{2n}8B^{n+1}2C^n 2D^{n-2}(8n-16)E$
$C_7: A^3 7B^2 4C$	5	$C_{4n+3}: A^{2n+1}7B^{n+1}4C^n D^{n-1}(8n-4)E$
$C_7: A^2 9B^2 2C$	5	$C_{4n+3}: A^{2n}9B^{n+1}2C^n D^{n-1}(8n-5)E$
$C_7: A10B^2 2C$	5	$C_{4n+3}: A^{2n-1}10B^{n+1}2C^n(8n-6)D$
$C_8: A^4 10B^2$	5	$C_{4n}: A^{2n}10B^n 2C^{n-2}(8n-12)D$
$C_{10}: A^4 8B^3 C^2 2D$	5	$C_{4n+2}: A^{2n}8B^{n+1}C^n 2D^{n-1}E^{n-2}(8n-14)F$
$C_8: A^4 9B^2 2C$	6	$C_{4n}: A^{2n}9B^n 2C^{n-1}D^{n-2}(8n-10)E$

Pencil $C_6: 8A^2 4B$

$\bar{C}$	Genus	Families of Involutions
$(C_6: 7A^2 4C)$	—	$C_{6n}: 7A^{2n}B^{2n-2}4C^n(8n-8)D$
$(C_6: 7A^2 2C)$	—	$C_{6n}: 7A^{2n}B^{2n-1}2C^n 2D^{n-1}(8n-7)E$
$C_1: A$	—	$C_{6n+1}: A^{2n+1}7B^{2n}4C^n(8n-4)D$
$C_1: 2B$	—	$C_{6n+1}: 8A^{2n}2B^{n+1}2C^n(8n-5)D$
$C_3: 7A$	1	$C_{6n+3}: 7A^{2n+1}B^{2n}4C^n(8n-2)D$
$C_3: 8A$	1	$C_{6n+8}: 8A^{2n+1}2B^n 2C^{n-1}(8n-5)D$
$C_3: 6A2C$	1	$C_{6n+3}: 6A^{2n+1}2B^{2n}2C^{n+1}2D^n(8n-3)E$
$C_4: A^2 7B2C$	2	$C_{6n+4}: A^{2n+2}7B^{2n+1}2C^{n+1}2D^n(8n-1)E$
$C_4: 8AB^2 2C$	2	$C_{6n+4}: 8A^{2n+1}B^{n+2}2C^{n+1}D^n(8n-2)E$
$C_4: 8A2B2C$	3	$C_{6n+4}: 8A^{2n+1}2B^{n+1}2C^{n+1}8nD$
$C_7: A^3 7B^2 2C^2$	3	$C_{6n+1}: A^{2n+1}7B^{2n}2C^{n+1}2D^{n-1}(8n-8)E$
$C_9: 8A^3 B^2$	3	$C_{6n+3}: 8A^{2n+1}B^{n+1}C^{n-1}2D^{n-1}(8n-7)E$

$\bar{C}$	Genus	Families of Involutions
$C_9: A^{2}7B^{3}C^{3}D$	3	$C_{6n+3}: A^{2n}7B^{2n+1}C^{n+2}D^{n}2E^{n-1}(8n-8)F$
$C_{12}: 7A^{4}B^{3}C^{4}D^{2}$	3	$C_{6n}: 7A^{2n}B^{2n-1}C^{n+2}D^{n}2E^{n-2}(8n-17)F$
$C_6: 6A^{2}2B4C$	4	$C_{6n}: 6A^{2n}2B^{2n-1}4C^{n}(8n-6)D$
$C_7: 8A^{2}3B^{2}$	4	$C_{6n+1}: 8A^{2n}3B^{n+1}C^{n-1}(8n-7)D$
$C_7: A^{3}7B^{2}C^{2}2D$	4	$C_{6n+1}: A^{2n+1}7B^{2n}C^{n+1}2D^{n}E^{n-1}(8n-6)F$
$C_{10}: 8A^{3}B^{4}2C^{2}$	4	$C_{6n+4}: 8A^{2n+1}B^{n+3}2C^{n+1}D^{n-1}(8n-8)E$
$C_{12}: 8A^{4}B^{3}C$	4	$C_{6n}: 8A^{2n}B^{n+1}C^{n-1}2D^{n-2}(8n-14)E$
$C_{15}: 8A^{5}B^{4}C^{2}$	4	$C_{6n+3}: 8A^{2n+1}B^{n+2}C^{n}2D^{n-2}(8n-15)E$
$C_{16}: A^{6}7B^{5}C^{5}D^{4}E$	4	$C_{6n+4}: A^{2n+2}7B^{2n+1}C^{n+3}D^{n+2}E^{n-1}F^{n-2}(8n-17)G$
$C_{10}: A^{4}7B^{3}C^{3}D^{2}E$	5	$C_{6n+4}: A^{2n+2}7B^{2n+1}C^{n+2}D^{n+1}E^{n}F^{n-1}(8n-5)G$
$C_{15}: 8A^{5}B^{4}2C$	5	$C_{6n+3}: 8A^{2n+1}B^{n+2}2C^{n-1}D^{n-2}(8n-13)E$
$C_{12}: 8A^{4}B^{2}2C$	5	$C_{6n}: 8A^{2n}B^{n}2C^{n-1}D^{n-2}(8n-10)E$

Pencil $C_7: A^{3}10B^{2}$

$\bar{C}$	Genus	Families of Involutions
$(C_7: A^{3}9B^{2})$	—	$C_{7n}: A^{3n}9B^{2n}C^{2n-2}(8n-8)D$
$C_1: A$	—	$C_{7n+1}: A^{3n+1}10B^{2n}(8n-4)C$
$C_3: A7B$	1	$C_{7n+3}: A^{3n+1}7B^{2n+1}3C^{2n}(8n-3)D$
$C_6: A^{2}7B^{2}2C$	2	$C_{7n+6}: A^{3n+2}7B^{2n+2}2C^{2n+1}D^{2n}(8n-2)E$
$C_6: A^{2}6B^{2}4C$	3	$C_{7n+6}: A^{3n+2}6B^{2n+2}4C^{2n+1}8nD$
$C_9: A^{3}7B^{3}C^{2}2D$	3	$C_{7n+2}: A^{3n}7B^{2n+1}C^{2n}2D^{2n-1}(8n-9)E$
$C_{12}: A^{4}7B^{4}C^{3}D^{2}E$	3	$C_{7n+2}: A^{3n+1}7B^{2n+2}C^{2n+1}D^{2n}E^{2n-1}(8n-10)F$
$C_7: A^{3}8B^{2}2C$	4	$C_{7n}: A^{3n}8B^{2n}2C^{2n-1}(8n-6)D$

Pencil $C_9: 8A^{3}2B^{2}C$

$\bar{C}$	Genus	Families of Involutions
$(C_9: 8A^{3}2BC)$	—	$C_{9n}: 8A^{3n}2B^{2n-1}C^{n}(8n-6)D$
$(C_9: 7A^{3}B^{2}2C^{2})$	—	$C_{9n}: 7A^{3n}B^{3n-1}2C^{2n}D^{n-1}(8n-6)E$
$C_3: 6A2CD$	1	$C_{9n+3}: 6A^{3n+1}2B^{3n}2C^{2n+1}D^{n+1}(8n-4)E$
$C_4: A^{2}7B2CD$	2	$C_{9n+4}: A^{3n+2}7B^{3n+1}2C^{2n+1}D^{n+1}(8n-2)E$
$C_6: 8A^{2}B$	2	$C_{9n+6}: 8A^{3n+2}B^{2n+1}C^{2n}D^{n}(8n-1)E$
$C_6: 6A^{2}2B2C^{2}$	2	$C_{9n+6}: 6A^{3n+2}2B^{3n+1}2C^{2n+2}D^{n}(8n-2)E$
$C_6: 7A^{2}B2CD$	3	$C_{9n+6}: 7A^{3n+2}B^{3n+1}2C^{2n+1}D^{n+1}8nE$
$C_7: A^{3}7B^{2}2C^{2}$	3	$C_{9n+7}: A^{3n+3}7B^{3n+2}2C^{2n+2}D^{n}8nE$
$C_9: 8A^{3}B^{2}D$	3	$C_{9n}: 8A^{3n}B^{2n}C^{2n-2}D^{n}(8n-8)E$
$C_{12}: 7A^{4}B^{2}C^{3}D^{4}$	3	$C_{9n+3}: 7A^{3n+1}B^{3n-1}C^{2n+1}D^{2n+2}E^{n-1}(8n-9)F$
$C_{12}: 8A^{4}B^{3}C^{2}$	3	$C_{9n+3}: 8A^{3n+1}B^{2n+1}C^{n+1}D^{2n-2}(8n-9)E$
$C_{12}: 8A^{4}B^{3}C$	4	$C_{9n+3}: 8A^{3n+1}B^{2n+1}C^{2n-1}D^{n-1}(8n-6)E$
$C_{10}: A^{4}7B^{3}C^{3}D^{2}E$	5	$C_{9n+1}: A^{3n+1}7B^{3n}C^{2n+1}D^{2n}E^{n}(8n-5)F$
$C_{15}: 8A^{5}B^{4}CD$	5	$C_{9n+6}: 8A^{3n+2}B^{2n+2}C^{2n-1}D^{n}(8n-5)E$

Pencil $C_{13}: A^{5}9B^{4}$

$\bar{C}$	Genus	Families of Involutions
$(C_{13}: A^{5}8B^{4}C^{3})$	—	$C_{13}: A^{6n}8B^{4n}C^{4n-1}(8n-5)D$
$C_6: A^{2}7B^{2}2C$	2	$C_{13n+6}: A^{5n+2}7B^{4n+2}2C^{4n+1}(8n-2)D$
$C_7: A^{3}9B^{2}$	3	$C_{13n+7}: A^{5n+3}9B^{4n+2}8nC$
$C_{12}: A^{4}7B^{4}C^{3}D^{2}$	3	$C_{13n+12}: A^{5n+4}7B^{4n+4}C^{4n+3}D^{4n+2}(8n-1)E$

3. *Pencils of Curves of Genus 3.* In order to proceed further with the problem, it is necessary to have a complete list of pencils of curves of genus 3, irreducible in type.

Let r_i denote the multiplicity of the i-th basis point, s the number of basis points, and n the order of the curve, where $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \geqq r_s > 0$. Therefore

$$(1) \qquad \sum_{i=1}^{s} r_i^2 = n^2,$$

since there are no variable intersections and since the genus, p, is 3, therefore, $(n-1)(n-2)/2 = r_i(r_i-1)/2 + 3$. Hence

$$(2) \qquad \sum_{i=1}^{s} r_i = 3n + 4.$$

Multiplying (2) by r_3 and substracting (1) from the product, we get

$$(3) \qquad r_3 \sum_{i=4}^{s} r_i - \sum_{i=r}^{s} r_i^2 = r_3(3n+4) - r_3r_1 - r_3r_2 - n^2 + r_1^2 + r_2^2$$
$$= n(3r_3 - n) - r_3(r_1 + r_2 - 4) + r_1^2 + r_2^2 \geqq 0,$$

since each term of the first summation is greater than or equal to the corresponding term of the second summation. And $n \geqq r_1 + r_2 + r_3$, because we are seeking irreducible pencils. Either $n > 3r_3$, or $r_1 = r_2 = r_3$, and therefore $n(3r_3 - n) \leqq (r_1 + r_2 + r_3)(2r_3 - r_1 - r_2)$; and, from (3), it follows

$$(r_1 + r_2 + r_3)(2r_3 - r_1 - r_2) - r_3(r_1 + r_2 - 4) + r_1^2 + r_2^2 \geqq 0,$$
$$(4) \qquad r_3(r_3 + 2) - r_1r_2 \geqq 0.$$
$$r_3 \leqq r_2, \qquad r_3 + 2 \geqq r_1.$$

Case I.

$$(5) \qquad r_3(r_3 + 2) - r_1r_2 = 0.$$

(6) Here $r_1 - 2 = r_2 = r_3 = R$, hence, from (3),

$$n(3R - n) - R(R + 2 + R - 4) + R^2 + 4R + 4 + R^2 \geqq 0.$$

For the equality, $n^2 - 3Rn - 6R - 4 = 0$, $n = 3R + 2$. Denoting the number of additional points of multiplicity R by p_0, the number of multiplicity $R-1$ by p_1, etc., (1) and (2) become

$$(R+2)^2 + 2R^2 + p_0R^2 + p_1(R-1)^2 + p_2(R-2)^2 + \cdots + p_{r-1} = 9R^2 + 12R + 4$$
$$R + 2 + 2R + p_0R + p_1(R-1) + p_2(R-2) + \cdots + p_{r-1} = 9R + 10.$$

Simplifying, we get

$$(A) \qquad p_0R^2 + p_1(R-1)^2 + p_2(R-2)^2 + \cdots + p_{r-1} = 6R^2 + 8R,$$

(B) $\quad p_0R + p_1(R-1) + p_2(R-2) + \cdots + p_{r-1} = 6R + 8.$

Multiplying (B) by $(R-1)$ and subtracting from (A), we get

(C) $\quad p_0R - p_2(R-2) - \cdots = 6R + 8.$

From (B), $p_0 \leqq (6+8)/R$, and from (C), $p_0 \geqq (6+8)/R$. Therefore $p_0 = (6+8)/R$ where both R and p_0 are integers.

$$\begin{array}{llll} R=1 & p_0=14 & s=17 & n=5 \\ R=2 & p_0=10 & s=13 & n=8 \\ R=4 & p_0=8 & s=11 & n=14 \\ R=8 & p_0=7 & s=10 & n=26 \end{array}$$

Hence we have the following pencils:

$$C_5\colon A^3 16B, \quad C_8\colon A^4 12B^2, \quad C_{14}\colon A^6 10B^4, \quad C_{26}\colon A^{10} 9B^8.$$

Case II.

(7) $$r_3(r_3+2) - r_1r_2 > 0.$$

Here $r_3 = r_1 - 1$ or r_1 and hence $r_2 = r_1 - 1$ or r_1.

If $r_1 - 1 = r_2 = r_3 = \cdots = r$, $n \geqq 3r + 1$. Equation (3) gives

$$n(3r-n) - r(2r-3) + 2r^2 + 2r + 1 > 0,$$
$$n^2 - 3nr - 5r - 1 < 0.$$

In order to establish an upper limit for n, let $n = 3r + 1 + x$.

$$9r^2 + 1 + x^2 + 6r + 6rx + 2x - 9r^2 - 3r - 3rx - 5r - 1 < 0,$$
$$x^2 - 2r + 3rx + 2x < 0, \quad x^2 + (3r+2)x - 2r < 0.$$

It is seen that x cannot be a positive integer, so $n \leqq 3r + 1$. Consider the possible cases when $n = 3r + 1$. From (1), we have

(8) $$(r+1)^2 + p_0r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 + 6r + 1,$$

where the p's have the same significance as above.

(9) $$(r+1) + p_0r + p_1(r-1) + p_2(r-2) + \cdots + p_{r+1} = 9r + 7.$$

From (8) and (9), we find

(10) $$p_0r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 8r^2 + 4r$$

and

(11) $\quad p_0 r + p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 8r + 6.$

(10) less $(r-1)$ times (11) gives

(12) $\quad p_0 r - p_2(r-2) - 2p_3(r-3) - \cdots - p_{r-1} = 6r + 6.$

From (10), $p_0 \leqq (8+4)/r$; and from (12), $p_0 \geqq (6+6)/r$. Also p_0 is an integer; hence $p_0 = 8$ or 9.

If $p_0 = 8$, (10) and (11) give

(13) $\quad p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 4r,$

(14) $\quad p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 6.$

The maximum value that the sum of the squares of the multiplicities of the remaining multiple points can have occurs when there is one point of multiplicity 6. Then $4r = 36$ and $r = 9$. Hence $r \leqq 9$. Also $r \geqq 2$ in order that some p be > 0.

The new solutions of (13) and (14) are obtained by allowing r to take on the integral values from 2 to 9 inclusive. When $r = 2$, (13) reduces to $p_1 = 4$, and (14) to $p_1 = 6$. So we see that r cannot equal 2, from (13),

When $r = 3$ $\quad 4p_1 + p_2 = 12$, from (13),

and $\quad 2p_1 + p_2 = 6$, from (14).

Hence $\quad 2p_1 = 6, \quad p_1 = 3, \quad n = 10.$

The pencil is C_{10}: $A^4 8B^3 3C^2$. Making use of (13) and (14), as above, with $r = 4, 5, 6, 7, 8, 9$, we find the following solutions:

$r = 4$; no solution.
$r = 5$; $p_1 = 1$, $p_2 = 0$, $p_3 = 1$, $n = 16$.
C_{16}: $A^6 8B^5 C^4 D^2$.
$r = 6$; no solution.
$r = 7$; no solution.
$r = 8$; no solution.
$r = 9$; $p_1 = p_2 = 0$, $p_3 = 1$, $n = 28$.
C_{28}: $A^{10} 8B^9 C^6$.

This completes the problem with $p_0 = 8$.

Putting $p_0 = 9$, reduces (10) and (11) to

$$p_1(r-1)^2 + p_2(r-2)^2 + p_3(r-3)^2 + \cdots + p_{r-1} = 4r - r^2 \quad \text{and}$$
$$p_1(r-1) + p_2(r-2) + p_3(r-3) + \cdots + p_{r-1} = 6 - r.$$

Since $4r - r^2 > 0$, $r < 4$. Also, $r > 1$. Proceeding as before

$$r = 2, \quad p_1 = 4, \quad n = 7.$$
$$C_7: A^3 9B^2 4C.$$
$$r = 3, \quad p_1 = 0, \quad n = 10.$$
$$C_{10}: A^4 9B^3 3C.$$

Suppose, as has been shown possible, that

$$r_1 = r_2 = r_3 = \cdots = r_s = r; \quad n \geq 3r.$$

Using the inequality sign, (3) becomes

$$n(3r - n) - r(2r - 4) + 2r^2 > 0; \text{ i. e.}$$
$$n^2 - 3rn - 4r < 0.$$

By an exactly analogous consideration, it is seen that

$$n = 3r \text{ or } n = 3r + 1$$

Consider $n = 3r$; (1) and (2) give

(15) $$p_0 r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 \text{ and}$$

(16) $$p_0 r + p_1(r-1) + p_2(r-2) = \cdots = p_{r-1} = 9r + 4.$$

(Equation (15) less $(r-1)$ times (16) gives

(17) $$p_0 r - p_2(r-2) - p_3(r-3) - \cdots - p_{r-1} = 5r + 4.$$

Hence, from (15), $p_0 \leq 9$; and, from (16), $p_0 \leq (9+4)/r$. From (17),

$$p_0 \geq (5+4)/r.$$

With $p_0 = 9$, (15) shows that all succeeding p's are 0; and (16) cannot be satisfied. So $p_0 = 6$, 7, or 8.

$$p_0 = 6; \text{ no solution.}$$
$$p_0 = 7, \quad (15) \text{ and } (16) \text{ reduce to}$$

(18) $$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 2r^2 \text{ and}$$

(19) $$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 2r + 4.$$

Proceeding as before, we find the following cases:

$r = 2$, $p_1 = 8$, $n = 6$. $C_6: 7A^2 8B$.
$r = 3$, $p_1 = 4$, $n = 9$. $C_9: 7A^3 4B^2 2C$.
$r = 4$, $p_1 = 3$, $p_2 = p_3 = 1$, $n = 12$. $C_{12}: 7A^4 3B^3 C^2 D$.

$r=5,\quad p_1=3,\quad p_2=p_3=0,\quad p_4=2,\quad n=15.\quad C_{15}: 7A^3 3B^4 2C;$
also
$p_1=p_2=2,\quad n=15.\quad C_{15}: 7A^5 2B^4 2C^3.$
$r=6;$ no solution.
$r=7,\quad p_1=2,\quad p_2=p_6=1,\quad p_3=p_4=p_5=0,\quad n=21.$
$C_{21}: 7A^7 2B^6 C^5 D.$
$r=8;$ no solution.
$r=9,\quad p_1=1,\quad p_2=2,\quad n=27.\quad C_{27}: 7A^9 B^8 2C^7.$

Equation (18) less $(r-2)$ times (19) gives

(20) $\quad p_1(r-1)-p_3(r-3)-\cdots-p_{r-1}=8.$ Hence $r\leq 9.$

With $p_0=8$, (15) and (16) become

(21) $\quad p_1(r-1)^2+p_2(r-2)^2+\cdots+p_{r-1}=r^2$ and

(22) $\quad p_1(r-1)+p_2(r-2)+\cdots+p_{r-1}=r+4.$

$p_1=0$ and $p_2=1$ reduce (21) and (22) to
$p_3(r-3)^2+\cdots+p_{r-1}=4r-4$ and $p_3(r-3)+\cdots+p_{r-1}=6,$

$$4r-4\leq 36,\quad r\leq 10,\quad r\geq 4.$$

When $4\leq r\leq 9$, there is no solution.

$$r=10,\quad p_3=0,\quad p_4=1,\quad n=30.\quad C_{30}: 8A^{10}B^8C^6.$$

Taking $p_1=1$, (21) and (22) become

$p_2(r-2)^2+\cdots+p_{r-1}=2r-1$ and $p_2(r-2)+\cdots+p_{r-1}=5$

$$2r-1\leq 25 \text{ or } r\leq 13.\quad \text{Also } r\geq 3.$$

$r=3,\quad p_2=5,\quad n=9.\quad C_9: 8A^3B^2 5C.$
$r=4,\quad p_2=1,\quad p_3=3,\quad n=12.\quad C_{12}: 8A^4B^3C^2 3D.$
$r=5,\quad p_4=1,\quad p_2=0,\quad p_3=2,\quad n=15.\quad C_{15}: 8A^5B^4 2C^2D.$
$r=6,\quad p_3=1,\quad p_5=2,\quad p_2=p_4=0,\quad n=18.\quad C_{18}: 8A^6B^5C^3 2D.$
$r=7,\quad p_2=p_3=p_6=0,\quad p_4=p_5=1,\quad n=21.\quad C_{21}: 8A^7B^6C^3D^2.$
$r=8;$ no solution.
$r=9,\quad p_2=p_3=p_4=p_6=p_7=0,$
$p_5=p_8=1,\quad n=27.\quad C_{27}: 8A^9B^8C^4D.$
$r=10$, 11, or 12; no solution.
$r=13,\quad p_2=p_3=p_4=p_5=p_6=p_7=p_9=p_{10}=p_{11}=p_{12}=0,$
$p_8=1,\quad n=39.\quad C_{39}: 8A^{13}B^{12}C^5.$

Now take $p_1=2.$ $\sum_{i=11}^{8} r_i=6-r$, whence $r<6.$ Also $\sum_{i=11}^{8} r_i^2=$

$4r - r^2 - 2$. So there are no solutions. If $p_1 > 2$, $\sum r_i^2$ and $\sum r_i$ show that no solutions are possible.

Finally, if $n = 3r + 1$ while $r_1 = r_2 = r_3 = r$, (1) and (2) become

$$(23) \quad p_0 r^2 + p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 9r^2 + 6r + 1$$

and

$$(24) \quad p_0 r + p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 9r + 7.$$

Equation (23) less $(r-1)$ times (24) yields

$$(25) \quad p_0 r - p_2(r-2) - \cdots - p_{r-1} = 8r + 8.$$

We see from (24) that $p_0 \leqq (9+7)/r$ and from (25) that $p_0 \geqq (8+8)/r$. So $p_0 = 10$, 11, 12, or 16.

Taking $p_0 = 10$, (23) and (24) become

$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - r^2$ and
$p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - r$; whence $r < 7$.
$r = 2$ or 3; no solutions.
$r = 4$, $p_1 = 1$, $n = 13$. $C_{13}: 10A^4B^3$.
$r = 5$; no solution.
$r = 6$, $p_1 = p_2 = p_3 = p_4 = 0$, $p_5 = 1$, $n = 19$. $C_{19}: 10A^6B$.

With $p_0 = 11$, we have from (23) and (24).

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - 2r^2$$

and $\quad p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} + 7 - 2r.$

So we note that r must be < 4.

$r = 2$; no solution.
$r = 3$, $p_1 = 0$, $p_2 = 1$, $n = 10$. $C_{10}: 11A^3B$.

When we put $p_0 = (12)$, (23) and (24) become

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 6r + 1 - 3r^2$$

and $\quad p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - 3r.$

The last equation demands that $r < 3$.

$$r = 2, \quad p_1 = 1, \quad n = 7. \quad C_7: 12A^2B.$$

with $p_0 = 16$, (23) and (24) become

$$p_1(r-1)^2 + p_2(r-2)^2 + \cdots + p_{r-1} = 32r - 16 - 7r^2$$

and $\quad p_1(r-1) + p_2(r-2) + \cdots + p_{r-1} = 7 - 7r$. Thus $p < 2$.

$p_1 = p_2 = \cdots = p_n = 0$, $n = 4$. $C_4: 16A$.

Here follows the set of arithmetically possible pencils of genus 3:

$C_4: 16A$
$C_5: A^3 16B$
$C_6: 7A^2 8B$
$C_7: 12A^2 B$
$C_7: A^3 9B^2 4C$
$C_8: A^4 12B^2$
$C_9: 7A^3 4B^2 2C$
$C_9: 8A^3 B^2 5C$
$C_{10}: A^4 8B^3 3C^2$
$C_{10}: A^4 9B^3 3C$
$C_{10}: 11A^3 B$
$C_{12}: 7A^4 3B^3 C^2 D$
$C_{12}: 8A^4 B^3 C^2 3D$
$C_{13}: 10A^4 B^3$
$C_{14}: A^6 10B^4$
$C_{15}: 7A^5 3B^4 2C$
$C_{15}: 7A^5 2B^4 2C^3$
$C_{15}: 8A^5 B^4 2C^2 D$
$C_{16}: A^6 8B^5 C^4 D^2$
$C_{18}: 8A^6 B^5 C^3 2D$
$C_{19}: 10A^6 B$
$C_{21}: 7A^7 2B^6 C^5 D$
$C_{21}: 8A^7 B^6 C^3 D^2$
$C_{26}: A^{10} 9B^8$
$C_{27}: 7A^9 B^8 2C^7$
$C_{27}: 8A^9 B^8 C^4 D$
$C_{28}: A^{10} 8B^9 C^6$
$C_{30}: 8A^{10} B^8 C^6$
$C_{39}: 8A^{13} B^{12} C^5$

Following the method employed with the pencils of curves of genus 2, the following families of involutions, based upon the pencil, $C_4: 16A$, have been found:

$\bar{C}$	Genus	Families of Involutions
$(C_4: 12A)$	—	$C_{4n}: 12A^n 4B^{n-1}(8n-8)C$
C_1	—	$C_{4n}: 16A^n(8n-3)B$
$C_3: 8A$	1	$C_{4n+3}: 8A^{n+1} 8B^n(8n-3)C$
$C_4: A^2 10B$	2	$C_{4n}: A^{n+1} 10B^n 5C^{n-1}(8n-10)D$
$C_5: A^3 13B$	3	$C_{4n+1}: A^{n+2} 13B^n 2C^{n-1}(8n-9)D$
$C_5: 2A^2 12B$	4	$C_{4n+1}: 2A^{n+1} 12B^n 2C^{n-1}(8n-7)D$
$C_6: 6A^2 8B$	4	$C_{4n+2}: 6A^{n+1} 8B^n 2C^{n-1}(8n-8)D$
$C_5: A^2 14B$	5	$C_{4n+1}: A^{n+1} 14B^n C^{n-1}(8n-5)D$
$C_6: 5A^2 10B$	5	$C_{4n+2}: 5A^{n+1} 10B^n C^{n-1}(8n-6)D$
$C_7: 10A^2 4B$	5	$C_{4n+3}: 10A^{n+1} 4B^n 2C^{n-1}(8n-7)D$
$C_7: A^3 7B^2 7C$	5	$C_{4n+3}: A^{n+2} 7B^{n+1} 7C^n D^{n-1}(8n-7)E$
$C_6: 4A^2 12B$	6	$C_{4n+2}: 4A^{n+1} 12B^n(8n-4)C$
$C_7: 9A^2 6B$	6	$C_{4n+3}: 9A^{n+1} 6B^n C^{n-1}(8n-5)D$
$C_7: 8A^2 8B$	7	$C_{4n+3}: 8A^{n+1} 8B^n(8n-3)C$

The remaining families of genus 3 are omitted chiefly on account of lack of space, but partly because the existence of some of the pencils has not been established. They can be found readily by the method used above.

4. *Geometric Considerations.* It remains to establish which of the pencils, listed in section 4, actually exist. In the cases in which there are at least three simple basis points, the two linearly independent members of the system are obtained by omitting three of the simple basis points, and determine by their intersection the three points omitted. In each of the other cases, a special investigation is necessary. As an example, the case of $C_{14}: A^6 10B^4$

is here considered. The remaining cases will be the subject of further investigations.

There are six linearly independent curves $C_7: A^3 8B^2$. Hence, for a general position of a point D, there are only three linearly independent curves $C_7: A^3 8B^2 D^2$. For special positions of D however, there are four linearly independent curves $C_7: A^3 8B^2 D^2$. Denote this web by $C_7: A^3 9B^2$. Through any point C in the plane pass three linearly independent curves $C_7: A^3 9B^2$; and one of these curves has a double point at C. Denote these curves by $C_7: A^3 9B^2 C^2$, $C_7': A^3 9B^2 C$, $C_7'': A^3 9B^2 C$, and let $C_7''': A^3 9B^2$ be a $C_7: A^3 9B^2$ not passing through C.

There are four linearly independent curves of order 14, $C_7 C_7'''$, $(C_7')^2$, $(C_7'')^2$, $C_7' C_7''$ having a double point at C. By a suitable linear combination of these, it is possible to get a C_{14} with a triple point at C. There were already two C_{14}'s having a triple point at C,—viz. $C_7 C_7'$ and $C_7 C_7''$. Since there are two degrees of freedom in the choice of C and of a linear combination of the three C_{14}'s, it is possible to find amongst them a C_{14} having a four-fold point at C. This C_{14}, together with $(C_7)^2$, define a pencil of curves $C_{14}: A^6 10B^4$.

PERSPECTIVITIES BETWEEN THE FUNDAMENTAL P-EDRA ASSOCIATED WITH THE ELLIPTIC NORM CURVE Q_p IN S_{p-1} WHERE P IS AN ODD PRIME.

By Bessie Irving Miller.

Associated with every elliptic norm curve of order p, Q_p in S_{p-1}, where p is an odd prime, is a set of $p+1$ fundamental p-edra. Each p-edron is composed of fixed points under an invariant subgroup, G_p, of the subgroup G_{p^2} of the G_{2p^2} of collineations which leave the members of the family $F(\infty^1)$ of Q_p's unaltered. The vertices and edges of the p-edra are the double points and constituent lines respectively of the degenerate members of the family, F.

The transformations on the coördinates of Q_p effected by the modular substitutions, S and T,

$$S: \begin{array}{l} \omega_1' = \omega_1 + \omega_2 \\ \omega_2' = \omega_2 \end{array} \qquad T: \begin{array}{l} \omega_1' = -\omega_2 \\ \omega_2' = \omega_1, \end{array}$$

are given by Klein and Fricke in the form

$$S: X_\alpha' = \epsilon^{-\alpha(p-\alpha)/2} X_\alpha, \qquad T: X_\alpha' = \sum_{\beta=0}^{p-1} \epsilon^{-\alpha\beta} X_\alpha,$$
$$(\epsilon = e^{2\pi i/p}).$$

These transformations at the most permute the p-edra.

There are $p+1$ p-edra, for the cyclic G_p's are generated by S_{01} and $S_{10}S_{01}{}^i$ $(1 = 0, \cdots, p-1)$, where S_{01} and S_{10} are of the form

$$S_{01}: u' = u + \omega_2/p, \qquad S_{10}: u' = u + \omega_1/p.$$

Since the p-edra are permuted by S and T the simplest method of securing them explicitly is to let P_∞ be the reference p-edron, secure another p-edron, P_0, by operating with T on P_∞, and then operate on P_0 with S^i $(i = 1, \cdots, p-1)$ to obtain the remaining p-edra, P_i $(i = 1, \cdots, p-1)$.

The vertices, V_α $(\alpha = 0, \cdots, p-1)$ of P_∞ have all coördinates equal to zero, except $X_\alpha = 1$. Since P_0 is obtained from P_∞ by operating with T, the coördinates of V_α of P_0 have the form

$$(1) \qquad X_\beta = \epsilon^{-\alpha\beta} \qquad (\alpha, \beta = 0, 1, \cdots, p-1).$$

Hence V_0 of P_0 has all of its coördinates equal to one, since $\alpha = 0$, and the first coördinate of every vertex is one since $\beta = 0$. The exponents of ϵ in X_β and $X_{p-\beta}$ are congruent (mod. p). Since the number of coördinates is odd,

it is always possible to isolate one and write $(p-1)/2$ relations between the others, taking them by pairs. The exponents of ϵ in X_β of V_α $(\alpha \neq 0)$, and $V_{p-\alpha}$, $(\alpha \neq 0$, are congruent (mod. p).

When S^i $(i=1, \cdots, p-1)$ is applied to P_0 to give P_i, then the coördinates of the vertices V_α or P_i are given by

$$(2) \qquad X' = \epsilon^{-\alpha\beta-\beta i(p-\beta)/2} X_\beta, \quad (i=1, \cdots, p-1; \beta = 0, 1, \cdots, p-1).$$

The first coördinate of every vertex equals one since $\beta = 0$. In every vertex except V_0 there is a second coördinate equal to one, for the condition that

$$X_\beta = 1$$

is that

$$-\alpha\beta - \beta i(p-\beta)/2 \equiv 0 \pmod{p},$$

or

$$\beta \equiv 0 \pmod{p} \text{ and } i\beta \equiv 2\alpha \pmod{p}.$$

The congruence

$$(3) \qquad i\beta \equiv 2\alpha \pmod{p}$$

for every i and every α can be solved for β, where $i=1, \cdots, p-1$, and $\alpha = 1, \cdots, p-1$.

In general two coördinates of V_α of P_i are equal, i. e.,

$$X_{\beta_1} = X_{\beta_2}$$

if

$$-\alpha\beta_1 - \beta_1 i(p-\beta)/2 \equiv -\alpha\beta_2 - \beta_2 i(p-\beta_2)/2 \pmod{p}$$

or

$$(\beta_2-\beta_1)\alpha + (ip/2)(\beta_2-\beta_1) \equiv (i/2)(\beta_2^2 - \beta_1^2) \pmod{p}.$$

Since $\beta_2 \neq \beta_1$,

$$(4) \qquad i(\beta_1+\beta_2) \equiv 2\alpha \pmod{p}$$

is the congruence which must be satisfied. The congruence (4) includes the congruence (3). Hence when β has been determined from (3), it is possible to secure β_1 and β_2 by taking numbers less than p whose sum equals β (mod. p). There are then $(p-1)/2$ pairs of equal coördinates in any one vertex V_i $(i=1, \cdots, p-1)$ of P_i since there are $(p-1)/2$ solutions of the congruence

$$\beta_1 + \beta_2 \equiv \beta \pmod{p}.$$

Moreover in every P_{i+1} there will be a vertex having the same pairs of equal ocördinates as that exhibited by any particular vertex of P_i, for if in the congruence (3′)

$$(3') \qquad (i+1)\beta \equiv 2\alpha \pmod{p}$$

β is fixed, there is one value of α which will satisfy (3′). Consequently the congruence (4) also can be satisfied.

By means of the relations established above it can be proved that if we choose a vertex of P_1 and write down the $(p-1)/2$ equalities between coördinates, there will be one vertex of each P_i, $(i=2,\cdots,p-1)$, whose coördinates satisfy the same relations. Hence there is a space $S_{(p-1)/2}$ lying on $p-1$ vertices of P_i, $(i=1,\cdots,p-1)$, one vertex from each P_i. Also one vertex of P_∞ will lie on the $S_{(p-1)/2}$. Since there are p vertices of P_1 there are p spaces $S_{(p-1)/2}$. But V_0 of P_0 is the point $(1, 1, 1, \cdots, 1)$. Therefore it lies on every space $S_{(p-1)/2}$ determined above. Hence the p-edra P_∞ and P_i, $(i=1,\cdots,p-1)$, are perspective with V_0 or P_0 as the center of perspectivity, where corresponding vertices of the p perspective p-edra are joined by spaces $S_{(p-1)/2}$.

Because of the group properties under which the p-edra are formed, symmetry requires that a relation true for one vertex of P_0 be matched by a similar relation for every other vertex of P_0. Moreover since the p-edra are interchangeable under the modular transformations, any one of the p-edra might have been isolated to furnish the centers of perspectivity.

Therefore the following theorem can be stated.

If the vertices of any one of the $p+1$ p-edra associated with Q_p in S_{p-1} where p is an odd prime, are chosen as centers of perspectivity, there are p sets of p spaces $S_{(p-1)/2}$ which join the corresponding vertices of the remaining p p-edra so that on each $S_{(p-1)/2}$ there is one vertex from each p-edron, and so that each set of p spaces $S_{(p-1)/2}$ lies on a vertex chosen as center of perspectivity. On every vertex chosen as center there is one set of p spaces $S_{(p-1)/2}$.

A simple but pretty illustration of the theory is given when $p=3$. Then the four flex triangles are the reference triangle, T_∞, and the triangles T_0, T_1, T_2.

T_0	T_1	T_2
$(1,1,1)$	$(1,\epsilon^{-1},\epsilon^{-1})$	$(1,\epsilon^{-2},\epsilon^{-2})$
$(1,\epsilon^{-1},\epsilon^{-2})$	$(1,\epsilon^{-2},1)$	$(1,1,\epsilon^{-1})$
$(1,\epsilon^{-2},\epsilon^{-1})$	$(1,1,\epsilon^{-2})$	$(1,\epsilon^{-1},1)$.

It is easy to see that if V_0 of T_0 is the center of perspectivity the lines joining corresponding vertices of T_1, T_2, T_∞ are

$$X_0 = X_1, \quad X_1 = X_2, \quad X_2 = X_0.$$

If V_1 of T_0 is the center of perspectivity the lines are

$$X_0 = \epsilon^{-2} X_1, \quad X_1 = \epsilon^{-2} X_2, \quad X_2 = \epsilon^{-2} X_0.$$

If V_2 of T_0 is the center of perspectivity the lines are

$$X_0 = \epsilon^{-1} X_1, \quad X_1 = \epsilon^{-1} X_2, \quad X_2 = \epsilon^{-1} X_0.$$

In an analogous fashion the equations of the planes joining corresponding vertices of the fundamental pentahedra associated with Q_5 in S_4, of the spaces S_3's joining corresponding vertices of the fundamental heptahedra associated with Q_7 in S_6, of the spaces S_5's joining corresponding vertices of the henahedra associated with Q_{11} in S_{10}, can be written down. If a vertex of P_0 is chosen to be the center of perspectivity, the choice of a vertex of P_1 then determines which coördinate is to be isolated and which $(p-1)/2$ relations between the remaining coördinates are to be used.

When $p = 3$, there are 3 lines on every vertex, and 4 vertices on every line. This is a different type of perspectivity from that usually mentioned, since there are 2 points on each line in excess of the 2 required to determine the line. Corresponding statements can be made for the other values of n under discussion, but the details are of little significance at present for $p > 7$ since little of the geometry of Q_p, $(p > 7)$, is as yet known.

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ON THE PROJECTIVE DIFFERENTIAL GEOMETRY OF CONJUGATE NETS.

By M. M. SLOTNICK.*

The analytic theory of the projective differential geometry of a conjugate net on a surface in ordinary space suffers a lack of elegance and symmetry in the two parameters. The method developed in the present paper is intended to overcome this disadvantage, and, in brief form, discusses the elementary theorems of conjugate nets. It will be noticed that, for the most part, the proofs are simple, direct, and not lacking in elegance as far as the symmetry in the parameters is concerned.

A special type of net—the net A—is introduced. It is hoped that these methods will yield further results on such nets as well as on the general and other special types of nets.

1. In a three-dimensional projective space, referred to a system of homogeneous point coördinates, we shall consider a non-developable surface S defined parametrically by the four equations

$$x_i = x_i(u, v), \qquad (i = 1, 2, 3, 4),$$

which we shall indicate by the single equation

$$x = x(u, v).$$

The parametric curves on this surface shall form a conjugate system; i. e., a net N.

Let z represent the coördinates of an arbitrary point on the axis of the net corresponding to the point of the net with the coördinates x. The four functions $x(u, v)$ can then be chosen, by using a suitable factor of proportionality,† to satisfy three equations of the forms

$$x_{uu} = Ax_u + Mx + ez, \tag{1.1}$$

$$x_{uv} = (\log a)_v x_u + (\log b)_u x_v, \tag{1.2}$$

$$x_{vv} = Bx_v + Nx + gz. \tag{1.3}$$

The asymptotic lines of S are defined by

$$edu^2 + gdv^2 = 0, \tag{1.4}$$

and, inasmuch as the surface is non-developable, $eg \neq 0$.

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† L. P. Eisenhart, *Transformations of Surfaces*, Princeton (1923), p. 72. Further references to this treatise will be indicated by *T. S.*

The coördinates of every point of the space can be written as linear combinations of x_u, x_v, x, z, since these four points are non-coplanar. For the present, we write

$$\begin{aligned} z_u &= \alpha_1 x_u + \alpha_2 x_v + \alpha_3 x + \alpha_4 z, \\ z_v &= \beta_1 x_u + \beta_2 x_v + \beta_3 x + \beta_4 z. \end{aligned} \tag{1. 5}$$

The coördinates of the focal points of the axis congruence are $z + \rho x$, where ρ is found to satisfy the equation

$$\rho^2 + \rho(\alpha_1 + \beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{1. 6}$$

We shall now choose the point z as the harmonic conjugate of x with respect to these two focal points, and, as a result,

$$\alpha_1 = -\beta_2 = r. \tag{1. 7}$$

The point z, so chosen, will be called the pivotal point * of the axis.

Here two cases may arise: either $r \not\equiv 0$, or $r \equiv 0$. If $r \not\equiv 0$ the coördinates z of equations (1. 5) may be replaced by rz. We shall then have $\alpha_1 \equiv -\beta_2 \equiv 1$. If, however, $r \equiv 0$, then $\alpha_1 \equiv -\beta_2 \equiv 0$. Both of these cases may be discussed simultaneously by introducing a symbol δ, whose significance is such that it is either equal to unity or equal to zero through the discussion.

2. Equations (1. 1), (1. 2), (1. 3), and (1. 5), in which $\alpha_1 \equiv -\beta_2 \equiv \delta$, form a completely integrable system. The conditions of integrability of the first three of these reduce to:

$$\begin{array}{ll} \text{a.} & A_v + e\beta_1 = (\log a)_{uv} + (\log a)_v(\log b)_u, \\ \text{b.} & A(\log b)_u + M - e\delta = (\log b)_{uu} + (\log b)^2_u, \\ \text{c.} & M_v + e\beta_3 = M(\log a)_v, \\ \text{d.} & \beta_4 = [\log(a/e)]_v, \end{array} \tag{2. 1}$$

and

$$\begin{array}{ll} \text{a.} & B(\log a)_v + N + g\delta = (\log a)_{vv} + (\log a)^2_v, \\ \text{b.} & B_u + g\alpha_2 = (\log b)_{uv} + (\log a)_v(\log b)_u, \\ \text{c.} & N_u + g\alpha_3 = N(\log b)_u, \\ \text{d.} & \alpha_4 = [\log(b/g)]_u. \end{array} \tag{2. 2}$$

The conditions of integrability for equations (1. 5), similarly, are:

$$\begin{array}{ll} \text{a.} & \delta(\log ae)_v + \beta_1[\log(b/g)]_u = \beta_{1u} + \beta_1 A + \beta_3, \\ \text{b.} & \delta(\log bg)_u - \alpha_2[\log(a/e)]_v = -\alpha_{2v} - \alpha_2 B - \alpha_3, \\ \text{c.} & \alpha_2 N + \alpha_{3v} + \beta_3[\log(b/g)]_u = \beta_1 M + \beta_{3u} + \alpha_3[\log(a/e)]_v, \\ \text{d.} & \alpha_{2}g + [\log(b/g)]_{uv} = \beta_1 e + [\log(a/e)]_{uv}. \end{array} \tag{2. 3}$$

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* This is the point to which Green calls attention, cf. G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 292, (21).

The last of these equations, taken with (2.1, a) and (2.2, b), indicate that we may write

i. e., $\quad A + [\log(b^2/g)]_u = \phi_u/\phi, \qquad B + [\log(a^2/e)]_v = \phi_v/\phi,$

2.4 $$A = [\log(\phi g/b^2]_u, \qquad B = [\log(\phi e/a^2)]_v,$$

where ϕ is some function of u and v.

Moreover, equations (2.1, b) and (2.2, a) may now be written:

2.5 $$\begin{aligned} M &= e\delta + (\log b)_{uu} + (\log b)_u[\log(b^3/\phi g)]_u, \\ N &= -g\delta + (\log a)_{vv} + (\log a)_v[\log(a^3/\phi e)]_v, \end{aligned}$$

and equations (2.1, a) and (2.2, b):

2.6 $$\begin{aligned} e\beta_1 &= [\log(ab^2/\phi g)]_{uv} + (\log a)_v(\log b)_u, \\ g\alpha_2 &= [\log(a^2b/\phi e)]_{uv} + (\log a)_v(\log b)_u. \end{aligned}$$

3. We shall now consider the conjugate net from the point of view of homogeneous tangential coördinates. The quadruple of ordered cofactors of a_1, a_2, a_3, a_4, respectively, in the determinant

$$|\, a\, b\, c\, d\, |$$

will be indicated by (bcd). Accordingly, the tangential coördinates of the surface S at the point x are

$$\xi = (x_u x_v x)/\psi,$$

where $1/\psi$ is, for the present, an arbitrary factor of proportionality.

Using the equations of the preceding section, we find that the tangential equation * of the net N is

3.1 $$\xi_{uv} = [\log(e\phi/a\psi)]_v\xi_u + [\log(g\phi/b\psi)]_u\xi_v,$$

where $\psi(\neq 0)$ is now chosen as an arbitrary solution of the equation

3.2 $$\begin{aligned} \psi_{uv} = {} & [\log(e\phi/a)]_v\psi_u + [\log(g\phi/b)]_u\psi_v \\ & + \{(\log ab)_{uv} + (\log a)_v(\log b)_u - [\log(g\phi/b)]_u[\log(e\phi/a)]_v\}\psi. \end{aligned}$$

The invariants of (3.1); i. e., the tangential invariants † of the net N, are found to be

3.3 $$\begin{aligned} H &= [\log(a^2b/\phi e)]_{uv} + (\log a)_v(\log b)_u, \\ K &= [\log(ab^2/\phi g)]_{uv} + (\log a)_v(\log b)_u. \end{aligned}$$

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* *T. S.*, p. 128.

† *T. S.*, p. 16 and p. 128.

These equations and equations (2. 6) indicate that

3. 4 $$\alpha_2 = \boldsymbol{H}/g, \qquad \beta_1 = \boldsymbol{K}/e.$$

As a first result, we note that the condition that *the u-curves be plane curves;* i. e., $(x\, x_u\, x_{uu}\, x_{uuu}) = 0$ *reduces to* $\boldsymbol{H} = 0$. Similarly, *the v-curves are plane if and only if* $\mathbf{K} = 0$.

4. The values of α_3 and β_3 are evaluated from (2. 2, c) and (2. 1, c), using (2. 5), thus obtaining:

4. 1 $$\begin{aligned} \alpha_3 &= -\delta[\log(b/g)]_u + (H/g)[\log(a^4H/\phi e)]_v - (\boldsymbol{H}/g)(\log a)_v, \\ \beta_3 &= \quad \delta[\log(a/e)]_v + (K/e)[\log(b^4K/\phi g)]_u - (\boldsymbol{K}/e)(\log b)_u. \end{aligned}$$

Here H and K are the point invariants of the net *; i. e.,

4. 2 $$H = -(\log a)_{uv} + (\log a)_v(\log b)_u, \quad K = -(\log b)_{uv} + (\log a)_v(\log b)_u.$$

The equations of compatibility (2. 3, a, b) can now be written

4. 3 $$\begin{aligned} &\text{a.} \quad \delta(\log e^2)_v + (\boldsymbol{K}/e)[\log(b^4e/\phi g^2\boldsymbol{K})]_v - (K/e)[\log(b^4K/\phi g)]_u = 0, \\ &\text{b.} \quad \delta(\log g^2)_u - (\boldsymbol{H}/g)[\log(a^4g/\phi e^2\boldsymbol{H})]_u + (H/g)[\log(a^4H/\phi e)]_v = 0. \end{aligned}$$

It is also well to point out that equations (4. 1) have the alternative forms:

4. 4 $$\begin{aligned} \alpha_3 &= -\delta[\log(bg)]_u + (\boldsymbol{H}/g)[\log(a^3g/\phi e^2\boldsymbol{H})]_v, \\ \beta_3 &= \quad \delta[\log(ae)]_v + (\boldsymbol{K}/e)[\log(b^3e/\phi g^2\boldsymbol{K})]_u. \end{aligned}$$

Finally, equations (1. 5), in their evaluated forms are:

4. 5 $$\begin{aligned} z_u &= \delta x_u + (\boldsymbol{H}/g)x_v + \alpha_3 x + [\log(b/g)]_u z, \\ z_v &= (\boldsymbol{K}/e)x_u - \delta x_v + \beta_3 x + [\log(a/e)]_v z. \end{aligned}$$

5. Consider an equation of the form (1. 2) and three functions e, g, and ϕ of u and v, satisfying the relations (4. 3) and (2. 3, c), where the functions involved are those defined in the preceding sections in terms of a, b, e, g, and ϕ. Equation (1. 2) and the equation

$$x_{vv} = (g/e)x_{uu} - (gA/e)x_u + Bx_v + (N - Mg/e)x$$

are then compatible.† Accordingly, a surface is defined in the three-dimen-

* *T. S.*, p. 58.

† The full proof of the facts here stated is contained in *T. S.*, pp. 100-103. The equations of compatibility in that text [p. 103, (22)] are fulfilled by virtue of our equations (4. 3) and (2. 3, c). To reconcile our functions with those of Eisenhart, it is to be noted that

$$r = g/e, \qquad a' = -gA/e, \qquad b' = B, \qquad c' = N - Mg/e,$$
$$a = a, \qquad b = b, \qquad c = 0.$$

sional space to within a projectivity, the parametric curves of which form a conjugate system, having (1.2) as its point equation. The pivotal point of the axis congruence associated with the net has coördinates of the form

$$z = (x_{uu} - Ax_u - Mx)/e = (x_{vv} - Bx_v - Nx)/g.$$

6. Let the tangential coördinates of the plane through a ray * of the net and the corresponding pivotal point be indicated by ζ. The point coördinates of the two Laplace transforms of the net are

6.1 $$x_{-1} = x_u - (\log b)_u x, \qquad x_1 = x_v - (\log a)_v x.$$

Accordingly,

6.2 $$\zeta = (x_{-1}\ x_1\ z)/\psi.$$

Using the relations of the preceding sections we find that

6.3 $$\begin{aligned} \zeta_u &= -\delta\xi_u - (H/g)\xi_v \\ &\quad + \{\delta[\log(\phi g^2/b\psi)]_u + (H/g)[\log(a^3H/\psi)]_v\}\xi + [\log(\phi/b\psi)]_u\zeta, \\ \zeta_v &= -(K/e)\xi_u + \delta\xi_v \\ &\quad + \{-\delta[\log(\phi e^2/a\psi)]_v + (K/e)[\log(b^3K/\psi)]_u\}\xi + [\log(\phi/a\psi)]_v\zeta. \end{aligned}$$

The equations dual to (1.1) and (1.3), the values of A, B, M, N being those given in § 2, are respectively:

6.4 $$\begin{aligned} \xi_{uu} &= [\log(eb^2\phi/\psi^2)]_u\xi_u \\ &\quad + \{e\delta + [\log(g\phi/b\psi]_{uu} + [\log(g\phi/b\psi]_u[\log(g\phi/eb^3)]_u\}\xi - e\zeta \end{aligned}$$

and

6.5 $$\begin{aligned} \xi_{vv} &= [\log(ga^2\phi/\psi^2)]_v\xi_v \\ &\quad + \{-g\delta + [\log(e\phi/a\psi)]_{vv} + [\log(e\phi/a\psi)]_v[\log(e\phi/ga^3)]_v\}\xi - g\zeta. \end{aligned}$$

All these equations and (3.1) are compatible by virtue of the relations (2.1), (2.2) and (2.3).

The plane which is the harmonic conjugate of the tangent plane to the surface with respect to the two focal planes of the corresponding ray of the net shall be termed the pivotal plane of the net. The tangential coördinates of these focal planes are of the form $\zeta + \rho\xi$ where ρ is determined from the fact that ξ, ζ, $(\zeta + \rho\xi)_u$, $(\zeta + \rho\xi)_v$ are linearly dependent. With the aid of (6.3), this condition reduces to

$$\rho^2 - (\delta + HK/eg) = 0.$$

* E. J. Wilczynski, *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 317.

Hence the coördinates of the pivotal plane are ζ; i. e., *the pivotal plane of a net passes through the corresponding pivotal point.*

7. The following equations are found to hold for the two Laplace transforms of the net:

$$\begin{aligned}
&\text{a.}\quad x_{-1u} = [\log(g\phi/b^3)]_u x_u + \{e\delta - (\log b)_u[\log(g\phi/b^3)]_u\}x + ez \\
&\qquad\quad = [\log(g\phi/b^3]_u x_{-1} + e(\delta x + z), \\
7.1\quad &\text{b.}\quad x_{-1v} = (\log a)_v x_u - (\log b)_{uv} x, \\
&\text{c.}\quad x_{-1uv} = \{K - H + (\log a)_v[\log(g\phi/b^2)]_u\}x_u \\
&\qquad\qquad + [M(\log a)_v - (\log b)_{uuv}]x + e(\log a)_v z \\
&\qquad\quad = (\log a)_v x_{-1u} + [\log(bK)]_u x_{-1v} + [K - H - (\log a)_v(\log K)_u]x_{-1}.
\end{aligned}$$

$$\begin{aligned}
&\text{a.}\quad x_{1u} = (\log b)_u x_v - (\log a)_{uv} x, \\
&\text{b.}\quad x_{1v} = [\log(e\phi/a^3)]_v x_v + \{-g\delta - (\log a)_v[\log(e\phi/a^3)]_v\}x + gz \\
7.2\quad &\qquad\quad = [\log(e\phi/a^3)]_v x_1 - g(\delta x - z), \\
&\text{c.}\quad x_{1uv} = \{H - K + (\log b)_u[\log(e\phi/a^2)]_v\}x_v \\
&\qquad\qquad + [N(\log b)_u - (\log a)_{uvv}]x + g(\log b)_u z \\
&\qquad\quad = [\log(aH)]_v x_{1u} + (\log b)_u x_{1v} + [H - K - (\log b)_u(\log H)_v]x_1.
\end{aligned}$$

The second part of equation (7. 1, a) indicates that the tangent to the u-curve of the minus first Laplace transform of the net intersects the axis in the point whose coördinates are $\delta x + z$. Similarly from (8. 2, b), the tangent to the v-curve of the first Laplace transform intersects the axis in the point $\delta x - z$. Thus, *the nets characterized by $\delta \equiv 0$ are those for which each tangent to a u-curve of the minus first Laplace transform of the net, the tangent to the v-curve of the first Laplace transform at the corresponding point and the corresponding axis of the net are concurrent in the pivotal point.**

For the general case ($\delta \equiv 1$), *the focal points of the axis of a net, the point in which the tangent to the u-curve of its minus first Laplace transform intersects that axis and the point in which the tangent to the v-curve of the first Laplace transform intersects the axis, the points of the axis which lie in the focal planes of the corresponding ray, are pairs of points in an involution, the double points of which are the point of the net and the corresponding pivotal point.*†

8. The equations of the developables of the axis congruence of the net, $(x\, z\, dx\, dz), = 0$ reduce to

* Such nets have been called *harmonic* by Wilczynski; cf. *American Journal of Mathematics*, Vol. 42 (1920), p. 215.

† Cf. the end of § 6, above, and also G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 306.

8.1 $$(\mathbf{H}/g)\,du^2 - 2\delta\, du\, dv - (\mathbf{K}/e)\,dv^2 = 0,$$

with the aid of equations (4.5).

The curves on the surface of the net defined by this equation have been called the *axis curves* associated with the net. They are obviously the curves of intersection of the surface with these developables.

Recalling equation (1.4) as defining the asymptotic lines of the surface, it follows that *the axis curves associated with a conjugate net form a conjugate system if and only if the net has equal tangential invariants. Tht axis congruence is then conjugate* * *to the surface. The axis curves coincide with the parametric net if and only if* $\delta \equiv 1$ *and the curves of the net are plane* †; *in this case the Laplace transforms of the net are developable* ‡; *the axis curves are indeterminate if and only if* $\delta \equiv 0$ *and both families of curves of the net are plane.*§

If the net has equal tangential invariants, it follows from (3.3) that

8.2 $$[\log(ag/be)]_{uv} = 0,$$

and conversely. Similarly, the net has equal point invariants if and only if

8.3 $$[\log(a/b)]_{uv} = 0.$$

Hence, *if a net has two of the following properties it has the third also: equal point invariants, equal tangential invariants, isothermal-conjugate.*¶

The equation of the developables of the ray congruence is obtained from the fact that for them

$$(x_1 \; x_{-1} \; dx_1 \; dx_{-1}) = 0.$$

We have the alternative method, however, dual to the method used above:

$$(\xi \; \zeta \; d\xi \; d\zeta) = 0.$$

These equations reduce to

8.4 $$(H/g)\,du^2 - 2\delta\, du\, dv - (K/e)\,dv^2 = 0.$$

The curves of the surface of the parametric net defined by this equation are called the *ray curves* of the net. The theorem of Wilczynski,‖ dual to the

* C. Guichard, *Annales de l'École Normale Supérieure*, 3°, Vol. 14 (1897), p. 478.

† Cf. end of § 3, above, and G. M. Green, *American Journal of Mathematics*, Vol. 38 (1916), p. 304.

‡ Cf. § 10.

§ Cf. footnote of G. M. Green, *l. c.*, p. 311.

¶ *T. S.*, p. 150.

‖ *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 318.

first part of the last theorem, follows from equations (8. 4) and (1. 4): *The ray curves associated with a conjugate net form a conjugate system if and only if the net has equal point invariants.*

Comparing equations (8. 1) and (8. 4), we note that *a necessary and sufficient condition that the axis curves and the ray curves associated with a net coincide is that each of the point invariants of the net be equal to the corresponding tangential invariant:*

$$H = \boldsymbol{H}, \qquad K = \boldsymbol{K}.$$

This last condition is equivalent, by virtue of (3. 3) and (4. 2), to

$$8.5 \qquad [\log(a^3b/\phi e)]_{uv} = 0, \qquad [\log(ab^3/\phi g)]_{uv} = 0.$$

We shall call a conjugate net for which the associated axis and ray curves coincide a net A. Equations (8. 2), (8. 3) and (8. 5) indicate that *if a net A has any one of the following four properties it has the other three also: equal point invariants, equal tangential invariants, isothermal-conjugate, the curves which are both the axis and ray curves of the net A form a conjugate system.*

A net A of this type is characterized by the fact that the four point and tangential invariants of it are all equal.

9. The focal points of the ray congruence associated with the net N have coördinates of the form $\sigma x_1 + \tau x_{-1}$, where σ/τ satisfies the relation:

$$9.1 \qquad \tau^2 K/g - 2\delta\sigma\tau - \sigma^2 H/e = 0.$$

The tangents to the ray curves, (8. 4), *will pass through these focal points if and only if $H = K$; i. e., if and only if the net has equal point invariants; the ray curves will then form a conjugate system.**

Guichard † has defined a conjugate net and a congruence of lines as *harmonic* to one another if the focal points of the lines of the congruence lie on the tangents to the curves of the net at corresponding points. Accordingly, the last result may be stated thus: *the ray curves associated with a net form a conjugate system and are harmonic to the ray congruence if and only if the net has equal point invariants.*

We also conclude readily, from equations (8. 1) and (9. 1) that *the tangents to the axis curves of a net pass through the focal points of the*

* *T. S.*, p. 124, problems 9 and 10.

† Guichard, *l. c.*

corresponding ray of the net if and only if each of the point invariants of the net is equal to the opposite tangential invariant:

$$H = \mathbf{K}, \qquad K = \mathbf{H}.$$

In view of (3.3) and (4.2), this condition reduces to

$$[\log(a^2b^2/\phi e)]_{uv} = 0, \qquad [\log(a^2b^2/\phi g)]_{uv} = 0. \tag{9.2}$$

Accordingly, *a necessary condition that the tangents to the axis curves of a net pass through the focal points of the corresponding ray is that the net be isothermal-conjugate.*

Again, *if the axis curves of a net A, which are also the ray curves of the net A, form a conjugate system, that conjugate system is harmonic to the ray congruence of the net A and is conjugate to the axis congruence of the net A.*

10. Using the methods of § 7 to compute x_{1uu} and x_{1vv}, we find that the asymptotic lines of the surface of the first Laplace transform of the net N are defined by

$$eH\,du^2 + g\mathbf{K}\,dv^2 = 0, \tag{10.1}$$

and those of the surface of the minus first Laplace transform by

$$e\mathbf{H}\,du^2 + gK\,dv^2 = 0 \tag{10.2}$$

We shall disregard the case $HK \equiv 0$ (cf. *T. S.*, p. 73).

It is well to note in the last two equations that if $\mathbf{H} = 0$ or $\mathbf{K} = 0$, the corresponding Laplace transform lies on a developable surface.

The developables of the ray congruence associated with the net, defined by equations (8.4), will intersect the surface of the first Laplace transform of the net in a conjugate system if and only if $K = \mathbf{K}$, and these will intersect the surface of the minus first Laplace transform in a conjugate net if and only if $H = \mathbf{H}$. This follows from equations (10.1) and (10.2). Thus, *a necessary and sufficient condition that the developables of the ray congruence of a conjugate net intersect the surfaces of the two Laplace transforms of the net in conjugate systems is that the net be a net A. The asymptotic lines on the two Laplace transforms will then correspond.*

The situation here is that *the surfaces of the two Laplace transforms are mapped upon one another by a fundamental transformation.** The axis congruence associated with the net will then be harmonic, in the sense of

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* *T. S.*, Chapter 2.

Guichard, to the conjugate systems on the surfaces of the Laplace transforms defined by (8.4). The two focal surfaces of the axis congruence of the net then carry Levy transforms of these conjugate systems.*

11. Tzitzéica † has defined an R net as a conjugate net such that the tangents to both families of curves of the net form W congruences.

Comparing equation (1.4) with equations (10.1) and (10.2), we conclude that *a necessary and sufficient condition that a conjugate net be a net R is that each of the point invariants of the net be equal to the opposite tangential invariant:*

$$H = \mathbf{K}, \qquad K = \mathbf{H}.$$

The similar conditions of § 9 indicate that *a necessary and sufficient condition that a net be a net R is that the tangents to the associated axis curves of the net pass through the focal points of the corresponding ray.* Moreover, *an R net is isothermal-conjugate.*‡

A theorem due to Demoulin § also follows at once: *If the tangents to the curves of either family of an isothermal-conjugate net form a W congruence, it is a net R.*

Again, *if a net R has any one of the following three properties, it has the other two also: equal point invariants, equal tangential invariants, it is a net A.*

Finally, *a net A which is isothermal-conjugate is a net R.*

* *l. c.* It is evident that the fundamental transformation existing between the surfaces of the Laplace transforms of a net A will be such that the product of its harmonic and its conjugate invariant will be unity (cf. Author, "A Contribution to the Theory of Fundamental Transformations of Surfaces, *Transactions of the American Mathematical Society*, Vol. 30, 18).

† *Comptes Rendus*, Vol. 152 (1911), p. 1077.

‡ Tzitzéica, *l. c.*

§ *Comptes Rendus*, Vol. 153 (1911), p. 592.

ADMISSIBLE NUMBERS IN THE THEORY OF GEOMETRICAL PROBABILITY.*

By A. H. COPELAND.

Geometry is concerned with non-denumerable aggregates of points. On the other hand, probability, from the point of view of its statistical definition, has an essentially denumerable character. Thus in geometrical probability we apply an analysis which is concerned with denumerable aggregates, to a subject which is concerned with non-denumerable aggregates. As a result we get certain inconsistencies in our assumptions. Fortunately these inconsistencies are not serious, and it is possible to obtain a set of assumptions which are both consistent and useful.†

The assumptions made in the case of a simple event have been shown consistent by proving the existence of admissible numbers.‡ In geometrical

* This paper was presented to the Society Sept. 7, 1928. It is based on a paper by the author, entitled, "Admissible Numbers in the Theory of Probability," *American Journal of Mathematics*, Vol. 50 (1928), pp. 535-552. The reader is referred to this memoir for definitions and notation.

† For other discussions of the foundations of the theory of probability see von Mises, "Grundlagen der Wahrscheinlichkeitsrechnung," *Mathematische Zeitschrift*, Vol. 5 (1919), pp. 52-99; Lomnicki, "Nouveaux fondéments du calcul des probabilités," *Fundamenta Mathematicae*, Vol. 4 (1923), pp. 34-71; Steinhaus, "Les probabilités dénombrables et leur rapport à la théorie de la mesure," *Fundamenta Mathematicae*, Vol. 4 (1923), pp. 286-310; Dodd, "Probability as Expressed by Asymptotic Limits of Pencils of Sequences," *Bulletin of the American Mathematical Society*, Vol. 36, No. 4 (1930), pp. 299-305; Borel, "Traité du calcul des probabilités," Chapitre I. The *nombres normaux* of Borel are members of the set, $A(1/2)$. That is, the set of admissible numbers includes the set of normal numbers as a sub set.

‡ See the author's memoir cited above. The statistical definition of probability has been criticised by T. C. Fry (*Probability and Its Engineering Uses*, pp. 88-91). Briefly Fry's argument is as follows. Let us suppose that the ratio, $p_n(x)$, of the number of successes to the number of trials of a given event, x, approaches the probability, p, as the number of trials is indefinitely increased. That is we assume that if we have given an arbitrary positive number, ϵ, we can find a number, N, such that $|p_n(x) - p| < \epsilon$ whenever $n \geqq N$. For definiteness let us choose $p = 1/2$, and $\epsilon = 1/4$. Then there must exist an N such that $|p_n(x) - 1/2| < 1/4$ whenever $n \geqq N$. Let us make N trials of the given event. If the result of the experiment is such that $p_n(x) > 1/2$, then since the trials are independent, there is a finite probability that the next N trials will all be successes. If $p_n(x) \leqq 1/2$, then it is possible for the next N trials all to be failures. In either case it is easily seen that $|p(x) - 1/2| \geqq 1/4$ when $n = 2N$. So far Fry's reasoning is correct. But he concludes that the statistical definition of probability is inconsistent with the postulate that the trials of a given event are all independent. This conclusion is not justified.

probability we are confronted with the problem of proving the existence of a set of related admissible numbers having the power of the continuum.

Let $\pi(E)$ be the probability that a point, P, of an n-dimensional continuum, belong to a given set, E. The function, $\pi(E)$, is necessarily additive and we will assume further that it is absolutely additive and absolutely continuous. These restrictions are satisfied in most of the cases that arise. Finally $0 \leqq \pi(E) \leqq 1$, and there exists a domain, Δ, such that $\pi(\Delta) = 1$. For example in the case of the normal law of error the domain, Δ, is the infinite interval, $(-\infty, +\infty)$, and the probability that the error of a given measurement be one of a set of values, E, is $\pi(E) = k/(\pi)^{1/2} \int_E e^{-k^2x^2}dx$.

We shall now investigate the admissibility of the event histories associated with the sets, E. Let $x(E)$ represent the event history associated with E. Then the function, $x(E)$, must satisfy the following restrictions.

(a) $x(E_1) \cdot x(E_2) = 0$ whenever $E_1 \cdot E_2 = 0$.

(b) If $E_1, E_2, E_3, \cdots$ is any sequence (finite or infinite) of mutually exclusive point sets, then the n-th digit of $x(E_1 + E_2 + E_3 + \cdots)$ is 1 if and only if the n-th digit of one of the numbers, $x(E_1)$, $x(E_2), \cdots$, is 1.

(c) $x(\Delta) = 1$.*

Condition (a) demands that events corresponding to mutually exclusive sets, be, themselves, mutually exclusive. Condition (b) demands that the function $x(E)$ be absolutely additive. It further specifies the mode of representation in certain cases where this representation is ambiguous. The interpretation of this condition is immediate.

The above conditions contain no reference to the function, $\pi(E)$, and no reference to the admissibility of the numbers, $x(E)$. We should expect, in fact, the further condition that every number, $x(E)$, must belong to the set, $A[\pi(E)]$. Moreover we should expect the numbers $(r_1/n)x(E_1)$, $(r_2/n)x(E_2), \cdots (r_k/n)x(E_k)$, to be independent for all sets, $E_1, E_2, \cdots E_k$,

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As stated above, I have proved that no such inconsistency can arise in the case of a simple event. Fry's reasoning merely shows that the choice of N depends upon x. That is, $p_n(x)$ does not approach its limit uniformly with regard to x. This has nothing to do with the existence of the limit for a given x.

One important conclusion can be drawn from Fry's reasoning. Namely that we can never in any physical situation, know the value of N. Hence in order to consider questions of consistency in the theory of probability we are forced to depend on some such device as that of admissible numbers.

* The number, 1, admits of two representations. For the sake of brevity we write, 1, whereas the representation to which we refer in this case is, .111, 111, 111, ...

and for all sets of positive integers, $r_1, r_2, \cdots, r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n. If these conditions, together with conditions, (a), (b), (c), could all be satisfied then the fundamental assumptions of geometrical probability would be consistent when applied to arbitrary sets, E. It turns out that these conditions cannot all be satisfied unless we confine ourselves to sets, E, whose frontier points are of measure zero. However, this restriction upon the point sets is so light that the assumptions of geometrical probability are satisfied in all of the interesting cases.

We are now in a position to state the fourth condition which we shall place upon the function, $x(E)$.

(d)
$$p[(r_1/n)x(E_1)\cdot(r_2/n)x(E_2)\cdots(r_k/n)x(E_k)] = \pi(E_1)\cdot\pi(E_2)\cdots\pi(E_k)$$

for all sets, $E_1, E_2, \cdots, E_k$, whose frontier points are of measure zero, and for all sets of positive integers, $r_1, r_2, \cdots r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n.

Condition (d) demands that every number, $x(E)$, belong to the set $A[\pi(E)]$, and that every set of numbers, $(r_1/n)x(E_1)$, $(r_2/n)x(E_2), \cdots (r_k/n)x(E_k)$, be independent provided the frontier points of the sets, $E, E_1, E_2, \cdots E_k$, are of measure zero. In order to see why we can include only those sets, E, whose frontier points are of measure zero, let us first investigate to what extent the function, $x(E)$, is restricted by conditions, (a), (b), (c).

THEOREM 1. *A necessary and sufficient condition that a function, $x(E)$, satisfy conditions, (a), (b), (c), is that there exist a denumerable set, $D: (P_1, P_2, P_3, \cdots)$ such that $D < \Delta$ and*

(1)
$$x(E) = \cdot\, \phi_E(P_1), \phi_E(P_2), \phi_E(P_3), \cdots$$

*where $\phi_E(P)$ is the fundamental function of the set, E.**

The condition is obviously sufficient so we shall concern ourselves with proving that it is necessary. We shall prove the theorem first for an n-dimensional continuum in which Δ is the region defined by the inequalities, $0 \leqq y_i < 1$, where $i = 1, 2, 3, \cdots n$ and where $y_1, y_2, \cdots, y_n$ are the coördinates of a point, P, in Δ. We shall cover Δ with a net consisting of an infinite set of lattices, $G_1, G_2, G_3, \cdots$. To construct the lattice, G_1, we

* See de la Vallée Poussin, "Sur l'intégral de Lebesgue," *Transactions of the American Mathematical Society* (1915).

decompose Δ into 2^n meshes, $m_{1,1}$, $m_{1,2}$, $m_{1,3}, \cdots m_{1,2^n}$, where the mesh, $m_{1,j}$, is the region, $a_{ij}/2 \leqq y_i < (a_{ij}+1)/2$ and where each a_{ij} has the value 1 or 0. The numbers, a_{ij}, are further defined so that the meshes, $m_{1,j}$, are non-overlapping and together include all of the points of Δ. To form the lattice, G_2, we decompose each of the meshes, $m_{1,j}$, into 2^n meshes, $m_{2,k}$, in the same manner in which Δ was decomposed to form the lattice, G_1. Thus G_2 contains 2^{2n} meshes. The lattices, G_3, G_4, etc. are formed in a similar manner. We shall assume that the meshes are so numbered that $m_{i-1,j}$ includes $m_{i,(j-1)2^n+1}$, $m_{i,(j-1)2^n+2}, \cdots m_{i,j2^n}$.

It follows from conditions, (a), (b), (c), that for a given i, one and only one of the numbers, $x(m_{i,j})$, has its first digit equal to one. Let this number be $x(m_{i,k_i})$. It follows from condition (a) that $m_{i+1,k_{i+1}} < m_{i,k_i}$. Thus if we let i become infinite we obtain a limiting set of the sequence, m_{i,k_i}. This set consists of a single point, P_1. By conditions (b), (c) we see that the first digit of $x(\Delta - P_1)$ must be zero and the first digit of $x(P_1)$ must be one. In a like manner we can find a point P_2 such that the second digit of $x(\Delta - P_2)$ is zero and the second digit of $x(P_2)$ is one. By continuing this process we obtain a denumerable set, D, consisting of the points, $P_1, P_2, P_3, \cdots$, and such that $x(\Delta - D) = 0$ and $x(D) = 1$. Thus $x(E)$ is given by equation, (1).

Next let us consider the case in which Δ includes all space. This case can be reduced to the above by means of the set of transformations, $T: y_i' = F(y_i)$, where $F(y)$ is monotone and continuous and such that $F(-\infty) = 0$ and $F(+\infty) = 1$ and where the inverse of $F(y)$ is continuous in the interval $0 < y' < 1$. In particular we can take $F(y) = 1/2 + y/2(1+y^2)^{1/2}$. If $x'(E') = x(E)$ where E' is the transformed set, E, then conditions, (a), (b), (c), are satisfied by $x'(E')$ provided they are satisfied by $x(E)$.* Hence this case is reduced to the above.

Finally if Δ is an arbitrary region then this case can be reduced to the preceding one by extending the definition of $x(E)$ outside of Δ by means of the equation, $x(E) = x(E \cdot \Delta)$.

Theorem 1 admits of a simple physical interpretation. The number, $x(E)$, represents the history of some imaginary event which succeeds or fails on the n-th trial according as P_n belongs or fails to belong to E. That is, P_n is the point obtained on the n-th trial of some imaginary physical ex-

* For sets, E' containing points with one or more coördinates equal to zero, $x'(E')$ is defined by the equation, $x'(E') = x'(E' \cdot \Delta')$, where Δ' is the region $0 \leqq y_i' < 1$. This extension of the definition is necessary since we shall exclude regions Δ having points at infinity.

periment. Thus we might have forseen the existance of the set, D, from physical considerations.

We can now see why it is possible to include only those sets whose frontier points are of zero measure. For let Δ be the unit interval, $0 \leqq y < 1$, and let $\pi(E)$ be the Lebesgue measure of the set, E. Then if $E = D + e$ where e is an interval of length $\epsilon < 1$, we have the equations, $\pi(E) = \epsilon$, $x(E) = 1$. Hence $x(E)$ is not an element of $A[\pi(E)]$. It will be observed that the frontier points of E are of measure, $1 - \epsilon > 0$.

I. Sets Whose Frontier Points Are of Measure Zero.

It turns out that our work will be greatly simplified if we replace condition (d) by a somewhat lighter condition which we shall refer to as (d′). Condition (d′) is the same as (d) except that we include only those sets, $E_1, E_2, \cdots E_k$, which consist of finite sums of meshes. We shall also have occasion to refer to a still lighter condition, (d″), which differs from (d′) only in that the meshes, $m_{i,j}$, which constitute the sets, $E_1, E_2, \cdots E_k$, are such that $i \leqq s$, where s is any given positive integer. We shall prove that if a function satisfies conditions (a), (b), (c) and (d′), it will automatically satisfy (d). In order to show this we shall make use of the following theorem.

Theorem 2. *If $E < \Delta$ where Δ is the region, $0 \leqq y_i < 1$, then a necessary and sufficient condition that the frontier points, f, of E be of measure zero, is that given any positive number, ϵ, there exist two sets, E_1 and E_2, such that $E_1 < E < E_2$ and $m(E_2 - E_1) < \epsilon$ and such that E_2 consists of a finite sum of meshes and E_1 is either null or else consists of a finite sum of meshes.*

The set, f, is closed and all of its points lie within or on the boundary of Δ. If $m(f) = 0$, then there exists an open set, 0, such that $f < 0$ and $m(0) < \epsilon$, where ϵ is an arbitrary positive number. If P is any point of $f \cdot \Delta$ then P lies in a mesh which in turn lies entirely within 0. If P is any other point of f then P lies on the boundary of a mesh which lies entirely within 0. It follows from the Heine-Borel theorem that there exists a set, $e < 0$, consisting of a finite sum of meshes and such that f is included in e plus those boundary points of e which are also boundary points of Δ.

Every interior point of E can be enclosed in a mesh which contains no points of f, and we have already obtained a law whereby the points of f can be enclosed in a finite sum of meshes plus certain boundary points of those meshes. Thus, applying the Heine-Borel theorem again we see that there exists a set, E_1, which is either null or else consists of a finite sum of

meshes no one of which includes points of f, and such that $E_1 + e > E$. Let $E_2 = E_1 + e$. Then $E_1 < E < E_2$ and $m(E_2 - E_1) = m(e) \leqq m(0) < \epsilon$. Moreover E_2 consists of a finite sum of meshes and E_1 is either null or else consists of a finite sum of meshes.

We shall now prove the converse. If we delete from E_1, the frontier points of E_1 we obtain a set 0_1, which differs from E_1 by a set of measure zero. Similarly if we add to E_2 the frontier points of E_2 we obtain a set F_2, which differs from E_2 by a set of measure zero. Then $F_2 - 0_1 > f$, and since the quantity, $m(F_2 - 0_1)$, can be made arbitrarily small by a proper choice of the sets, E_1 and E_2, it follows that $m(f) = 0$.

THEOREM 3. *If $\pi(E)$ is an absolutely additive absolutely continuous function defined in $\Delta: 0 \leqq y_i < 1$, and if $x(E)$ is a function which satisfies conditions (a), (b), (c), (d') with respect to $\pi(E)$, then $x(E)$ satisfies conditions (a), (b), (c), (d), with respect to $\pi(E)$.*

If $E_1, E_2, \cdots E_k$ are any sets such that they all lie in Δ and their frontier points are of measure zero, then given any positive number ϵ we can find sets, $E_1', E_2', \cdots E_k', E_1'', E_2'', \cdots E_k''$, consisting of finite sums of meshes and such that $E_i' < E_i < E_i''$ and

$$\pi(E_1'') \cdot \pi(E_2'') \cdots \pi(E_k'') - \epsilon/2$$
$$\leqq \pi(E_1) \cdot \pi(E_2) \cdots \pi(E) \leqq \pi(E_1') \cdots \pi(E_k') + \epsilon/2.$$

Since $x(E)$ satisfies condition (d′) it follows that if we have given any set of positive integers, $r_1, r_2, \cdots r_k, n$, such that the numbers, r_i, are all distinct and less than or equal to n, then we can select a number, s_0, such that

$$\pi(E_1') \cdot \pi(E_2') \cdots \pi(E_k') - \epsilon/2 \leqq p_s\left[(r_1/n)x(E_1') \cdots (r_k/n)x(E_k')\right]$$

and

$$p_s\left[(r_1/n)x(E_1'') \cdots\right] \leqq \pi(E_1'') \cdots \pi(E_k'') + \epsilon/2$$

whenever $s > s_0$. Moreover since $x(E)$ satisfies conditions (a), (b), (c) it is defined by equation (1) and hence

$$p_s[(r_1/n)x(E_1') \cdots] \leqq p_s[(r_1/n)x(E_1) \cdots] \leqq p_s[(r_1/n)x(E_1'') \cdots].$$

Combining these inequalities we get the relation

$$\pi(E_1) \cdots \pi(E_k) - \epsilon < p_s[(r_1/n)x(E_1) \cdots] < \pi(E_1) \cdots \pi(E_k) + \epsilon$$

whenever $s > s_0$. Therefore $x(E)$ satisfies (a), (b), (c), (d).

II. Admissibly Ordered Sets.

We shall say that a denumerable set, D, is admissibly ordered with respect to a given function, $\pi(E)$, provided the corresponding function, $x(E)$, satisfies conditions, (a), (b), (c), and (d).

Theorem 4. *There exists a set, D, which is admissibly ordered with respect to the function, $m(E)$, (the Lebesgue measure of E) defined in the unit interval, $\Delta: 0 \leqq y < 1$.*

We shall show that conditions, (a), (b), (c), (d″), can be satisfied. We shall then prove that the restriction, $i \leqq s$, of condition (d″), can be removed.

Let us define a function, $X_s(E)$, as follows. Let X be any member of the set, $A(1/2)$ and let $X_s(m_{11}) = (1/s)X$ and $X_s(m_{12}) = 1 - (1/s)X$. In general let

$$X_s(m_{i,2j-1}) = X_s(m_{i-1,j}) \cdot (i/s)X$$

and

$$X_s(m_{i,2j}) = X_s(m_{i-1,j}) \cdot [1 - (i/s)X]$$

where $1 < i \leqq s$. Then $X_s(m_{i,j}) \cdot X_s(m_{i,j''}) = 0$ if $j \neq j'$ and

$$X_s(m_{i-1,j}) = X_s(m_{i,2j-1}) + X_s(m_{i,2j}).$$

Hence we will introduce no contradiction in our notation if we define $X_s(\Sigma m_{i,j})$ to be equal to $\Sigma X_s(m_{i,j})$ where the meshes, $m_{i,j}$, which appear in the summation, are mutually exclusive. Thus conditions, (a), (b), are satisfied by the function, $X_s(E)$, for all sets, E, consisting of sums of meshes, $m_{i,j}$ $(i \leqq s)$. Moreover condition (d″) is satisfied since every number, $X_s(m_{i,j})$, can be written as the product of i numbers each of which is of the form,* $(k/s)X$ or $1 - (k/s)X$.

Corresponding to the function, $X_s(E)$, we can define a set, $D_s: (P_1^s, P_2^s, P_3^s, \cdots)$, as follows. The point, P_n^s, is an arbitrary point of that mesh, $m_{s,j}$, for which the n-th digit of $X_s(m_{s,j})$ is equal to 1. The function, $X_s(E)$, can now be defined for all sets, E, by means of equation 1. This function satisfies conditions, (a), (b), (c).

To remove the restriction, $i \leqq s$, we shall define a new set, $D: (P_1, P_2, P_3, \cdots)$. Points P_{ν_s+1} to $P_{\nu_{s+1}}$ of D are the same as points, P_1^s to $P_{N_s}^s$, of D_s, where $\nu_s = N_1 + N_2 + \cdots N_{s-1}$. We shall show that the integers, $N_1, N_2, \cdots$, can be chosen so that the set, D, will be admissibly ordered.

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* See Theorem 16 of the memoir previously cited.

Let $\epsilon_1, \epsilon_2, \cdots \epsilon_s, \cdots$ be a decreasing sequence of positive numbers having the limit zero. We can choose two sets of integers, $M_1, M_2, M_3, \cdots M_s, \cdots$ and $N_1, N_2, \cdots$, such that

$$\text{(e)}\quad | p_N[(r_1/n)X_{s+1}(E_1)\cdot(r_2/n)X_{s+1}(E_2)\cdots(r_k/n)X_{s+1}(E_k)] \\ -m(E_1)\cdot m(E_2)\cdots m(E_k) \mid < \epsilon_s/3 \quad \text{if} \quad N \geqq M_s/n$$

$$\text{(f)}\quad | p_N[(r_1/n)X_s(E_1)\cdot(r_2/n)X_s(E_2)\cdots(r_k/n)X_s(E_k) \\ -m(E_1)m(E_2)\cdots m(E_k) \mid + (\nu_s + M_s)/N_s < \epsilon_s/3 \text{ if } N \geqq N_s/n.$$

The numbers, M_s, and N_s, are so chosen that conditions, (e) and (f), hold for every set of positive integers, $r_1, r_2, \cdots r_k, n$, such that $n \leqq s$ and the numbers, r_i, are all distinct and less than or equal to n, and for all sets, $E_1, E_2, \cdots E_k$, consisting of sums of meshes, $m_{i,j}$, such that $i \leqq s$. At the same time the numbers, N_s, are so chosen that ν_s/n is an integer if $n \leqq s$.

Since digits $\nu_s + 1$ to ν_{s+1} of $x(E)$ are the same as digits 1 to N_s of $X_s(E)$ it follows that

$$\text{(g)}\quad | p_N[(r_1/n)x(E_1)\cdots(r_k/n)x(E_k)] \\ -m(E_1)\cdot m(E_2)\cdots m(E_k) \mid < \epsilon_s \quad \text{if} \quad \nu_{s+1}/n \leqq N \leqq \nu_{s+2}/n.^*$$

Moreover the restrictions on i and n are no longer necessary, for if we select first the sets, $E_1, E_2, \cdots E_k$, consisting of finite sums of meshes, $m_{i,j}$, and next the numbers, $r_1, r_2, \cdots r_k, n$, then condition (g) holds for every s which is at the same time greater than n and greater than the largest subscript, i. Hence

$$p[(r_1/n)x(E_1)\cdots(r_k/n)x(E_k)] = m(E_1)\cdot m(E_2)\cdots m(E_k).$$

It follows from theorem 3 that D is admissibly ordered.

THEOREM 5. *Given a denumerable set, D, which is everywhere dense in the domain, $\Delta: 0 \leqq y_i < 1$, and given an absolutely additive absolutely continuous function, $\pi(E)$, such that $\pi(\Delta) = 1$ and $\pi(E) > 0$ if $m(E) > 0$ (where $E < \Delta$), then the set, D, can be admissibly ordered with respect to the function, $\pi(E)$.*

Let D consist of the points, $P_1, P_2, P_3, \cdots$, and let $D'': P_1'', P_2'', \cdots$ be the reordered set, D. We have to show that the reordering can be accomplished in such a manner that the function, $x(E) = \cdot\, \phi_E(P_1''), \phi_E(P_2'')\cdots$, satisfies condition (d').

Let Δ' be the domain, $0 \leqq y < 1$, and let $D': P_1', P_2' \cdots$ be a set which is admissibly ordered with respect to the function, $m(E')$. We shall set up

* Compare theorem 11 of the memoir cited.

a correspondence between the meshes $m_{i,j}$, and a set of half open intervals, $m'_{i,j}$, which are included in Δ'. The interval, $m'_{i,j}$, is given by the inequalities,

$$\pi(m_{i1}+m_{i2}+\cdots m_{i,j-1}) \leqq y < \pi(m_{i1}+\cdots m_{i,j}), \quad \text{if} \quad j>1.$$

The interval, $m'_{i,1}$, is given by the inequalities, $0 \leqq y < \pi(m_{i,1})$. Then the interval, $m'_{i-1,j}$, will contain the intervals, $m'_{i,(j-1)2^n+1}$, $m'_{i,(j-1)2^n+2}, \cdots, m'_{i,j2^n}$. Thus the intervals will cover Δ' in the same manner that the meshes cover Δ.

Every point of D' is included in one of the intervals, $m'_{1,j}$. Let m'_{1,j_1} be the interval which includes the point, P_1'. We shall select from those points of D which lie in m_{1,j_1}, that one whose subscript is least. We shall relabel this point, P_1''. We shall continue this process with the points, $P_2', P_3', \cdots$, setting up a correspondence in each case with that point of D which lies in the proper interval and which has the least subscript of those points not already assigned. We finally reach a point, P_{n_1}', such that at least one of the points, $P_1', P_2', \cdots P_{n_1}'$, lies in each of the intervals, $m'_{1,j}$. One of the n_1 relabeled points must be the point, P_1.

We shall select a number, n_2, so that at least one of the points, P'_{n_1+1}, $P'_{n_1+2}, \cdots P'_{n_2}$, lies in each of the intervals, $m_{2,j}$. Using these intervals we shall relabel $n_2 - n_1$ more of the points of D in the manner described above. We shall call the relabeled points, $P''_{n_1+1}, P''_{n_1+2}, \cdots P''_{n_2}$. We have now relabeled the point, P_2. This process is continued indefinitely. By the time the assignment of points has been completed for the k-th lattice, the point, P_k, has been relabeled.

We shall now show that the reordering of D has been accomplished in such a manner that $x(E)$ satisfies condition (d′) and hence satisfies (d). Let $E_1, E_2, \cdots E_k$ be any sets consisting of finite sums of meshes. Let $E_1', E_2', \cdots E_k'$ be corresponding linear sets, the correspondence being defined in terms of the correspondence which we have already established between meshes and half open intervals. Then at most a finite number of the digits of $x(E_i)$ differ from the corresponding digits of

$$x'(E_i') = \cdot\, \phi_{E_i''}(P_1'),\ \phi_{E_i'}(P_2') \cdots.$$

Therefore

$$\begin{aligned} p[(r_1/n)x(E_1)\cdot(r_2/n)x(E_2)\cdots(r_k/n)x(E_k)] \\ = p[(r_1/n)x'(E_1')\cdot(r_2/n)x'(E_2')\cdots] \\ = \pi(E_1)\cdot\ \pi(E_2)\cdots\pi(E_k). \end{aligned}$$

Hence D has been admissibly ordered with respect to $\pi(E)$.

In theorem 5, the restriction, $\pi(E) > 0$ if $m(E) > 0$, was made in order that D could be reordered without leaving out any of its points. Let us see

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to what extent this restriction could be removed. Let Δ be the region $0 \leqq y_i < 1$, and let us define a set, Δ_0, by means of its complement with respect to Δ. The set $C\Delta_0$ will consist of all of the meshes, $m_{i,j}$, for which $\pi(m_{i,j}) = 0$. It is easily seen that $\pi(C\Delta_0) = 0$ and $\pi(\Delta_0) = 1$. Let D be a denumerable set such that $D < \Delta_0$ and every point of Δ_0 is a limit point of points of D. Obviously D can be admissibly ordered with respect to $\pi(E)$. We shall call D the skeleton set of the function, $\pi(E)$.

Next let us consider an arbitrary absolutely additive absolutely continuous non-negative function, $\pi(E)$, such that $\pi(\Delta) = 1$. The most general case is that in which Δ includes all space. Let us apply transformations, T, to the sets, E, and let $\pi'(E')$ be the transformed function $\pi(E)$. We can define a skeleton set, D', for the function, $\pi'(E')$, the inverse of the transformations, T, carry D' into a set, D, which we shall call the skeleton set of the function $\pi(E)$. The following theorem is now obvious.

THEOREM 6. *If $\pi(E)$ is an absolutely additive absolutely continuous non-negative function such that $\pi(\Delta) = 1$, where Δ includes all space, then the skeleton set, D, of $\pi(E)$ can be admissibly ordered with respect to $\pi(E)$.*

The admissibly ordered skeleton set, D, characterizes the function, $\pi(E)$. In fact we have the equation,

$$\pi(E) = \lim_{n \to \infty} \sum_{k=1}^{n} \phi_E(P_k)/n$$

which holds for all sets whose frontier points are of zero measure.

We have now proved that the fundamental assumptions of geometrical probability are valid when applied to sets whose frontier points are of measure zero, but that they lead to inconsistencies if applied to arbitrary sets.

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CONTINUOUS CURVES WITHOUT LOCAL SEPARATING POINTS.*

By G. T. Whyburn.

1. In this paper it will be shown that every pair of points a and b of a continuous curve M which has no local separating point lie together on a subcontinuum T of M which is the sum of c (= the power of the continuum) independent simple continuous arcs from a to b. It follows at once from this result that every continuous curve which has no local separating point contains continua that are not locally connected or, in other words, that every continuous curve all of whose subcontinua are continuous curves has local separating points.

We use the term *continuous curve* to designate any locally compact, metric, separable, connected and locally connected space. Any connected open subset of such a space is called a *region;* and a point which is a cut point of at least one region in the space is called a *local separating point* of the space.

2. Lemma. *Let R be any compact region in a continuous curve M, and let N be a closed subset of $M - R$ such that $\bar{R} \cdot N$ is totally disconnected. Then there exists a compact region G containing R but containing no point of N and such that* (1) $\bar{G} \cdot N = \bar{R} \cdot N$, (2) *$G + \bar{R} \cdot N$ contains a compact continuous curve H which contains $\bar{R} \cdot N$ and is such that $H - \bar{R} \cdot N$ is connected and contains R, and* (3) *each point of $\bar{R} \cdot N$ is accessible from $H - \bar{R} \cdot N$ and hence also from G.*

Proof. Let K_1 denote the set of all points of $\bar{R}$ at a distance > 1 from the point set $\bar{R} \cdot N$; and for each integer $n > 1$, let K_n denote the set of all points x of $\bar{R}$ such that $1/n \leq \rho(x, \bar{R} \cdot N) \leq 1/(n-1)$.

A simple application of the Borel Theorem proves the existence, for each positive integer n, of a finite number of compact continua C_1^n, C_2^n, $\cdots$, C_m^n each containing a point of R and whose sum C^n contains K_n in its interior (rel. M) but contains no point of N and no point whose distance from K_n is greater than $1/4n$. For each i, $1 < i \leq m$, let t_i be an arc in R joining a point C_i^n to a point of C_1^n. Add all these arcs t_i to C^n and call the point set thus obtained D_n. Then D_n is a compact continuum which contains K_n but contains neither a point of N nor any point of $M - R$ whose distance from K_n is greater than $1/4n$. By a theorem due to Ayres

* Presented to the American Mathematical Society, Sept. 10, 1930.

and the author,* M contains, for each n, a compact continuous curve H_n containing D_n but which contains neither a point of N nor any point of $M - R$ whose distance from K_n is greater than $1/2n$.

Let $H_0 = \sum_1^\infty H_n$, and let $H = H_0 + \bar{R} \cdot N$. For each n, let G_n be a compact region containing H_n but containing no points or boundary points in N and containing no point whose distance from H_n is greater than $1/2n$. Let $G = \sum_1^\infty G_n$. Then the sets G and H have the desired properties.

2. Theorem. *If a and b are any two points of a continuous curve M having no local separating point, then there exists a subcontinuum T of M such that*

$$T = \sum_{0 \leq x \leq 1} axb,$$

where, for each x, axb is an arc from a to b and where, for $x \neq y$, $0 \leq x, y \leq 1$, $axb \cdot ayb = a + b$.

We shall first prove the existence of a continuum T satisfying all the conditions except that the sets axb, $(0 \leq x \leq 1)$, are compact continua but not necessarily simple arcs; and then we shall give the modifications in this argument which are necessary to insure that the continua $[axb]$ will be arcs from a to b.

Since no point separates a and b in M, there exist † in M two independent ‡ arcs T_0 and T_1 from a to b. There exist in M two compact regions R_0 and R_1 containing the open arcs $T_0 - (a + b)$ and $T_1 - (a + b)$ respectively and such that $\bar{R}_0 \cdot \bar{R}_1 = a + b$. Now, applying the lemma in turn to the regions R_0 and R_1, using for the closed set N in the lemma first the set $\bar{R}_1$ and then the set $\bar{G}_0$, we obtain two compact regions G_0 and G_1 such that (1) $\bar{G}_0 \cdot \bar{G}_1 = a + b$, (2) $\bar{G}_0$ and $\bar{G}_1$ contain continuous curves H_0 and H_1 respectively both of which contain $a + b$ and such that $H_0 - (a + b)$ and $H_1 - (a + b)$ are connected sets which contain R_0 and R_1 respectively.

Since $H_0 \supset R_0$, clearly no point can separate a and b in the continuous curve H_0. Hence there exist in H_0 two independent arcs T_{00} and T_{01} from a to b. There exist in G_0 two regions R_{00} and R_{01} containing the open arcs $T_{00} - (a + b)$ and $T_{01} - (a + b)$ respectively and such that $\bar{R}_{00} + \bar{R}_{01} \subset G_0 + a + b$ and $\bar{R}_{00} \cdot \bar{R}_{01} = a + b$. Then, applying the lemma just as in

* See *Bulletin of the American Mathematical Society*, Vol. 34 (1928), p. 350.

† See the author's paper in the *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 308, Theorem III, and a paper by W. L. Ayres in this Journal, Vol. 51 (1929), p. 590, where the author's theorem is extended to the more general space required in the present application.

‡ Two arcs are said to be independent if they have at most their end points in common.

the preceeding paragraph, we obtain regions G_{00} and G_{01} in R_0 and continuous curves H_{00} and H_{01} such that, $\bar{G}_{00}\cdot\bar{G}_{01}=(a+b)$, $R_{00}\subset H_{00}-(a+b)\subset H_{00}\subset\bar{G}_{00}$, and $R_{01}\subset H_{01}-(a+b)\subset H_{01}\subset\bar{G}_{01}$. Similarly in H_1 we get the arcs T_{10} and T_{11} from a to b and in G_1 we get regions R_{10}, R_{11}, G_{10} and G_{11} and continuous curves H_{10} and H_{11} satisfying similar conditions. Continue this process indefinitely.

At each stage n, let G_n denote the sum of the 2^n closed regions $\bar{G}_{a_1,a_2\ldots a_n}$ constructed at that stage, i. e., let

$$G_n=\sum_{a_i=0,1}\bar{G}_{a_1,a_2,\ldots a_n}.$$

Let $T=\prod_1^\infty G_n$. Then T is the desired continuum. For let x be any real number, $0\leq x\leq 1$. Write $x=.a_1a_2a_3\cdots$, where for each i, a_i is either 0 or 1.* Now the point set $axb=\prod_1^\infty\bar{G}_{a_1a_2\ldots a_i}$ is a subcontinuum of M; and if x and y are distinct numbers between 0 and 1, it is readily seen that $axb\cdot ayb=a+b$.

3. I shall now indicate the modifications necessary in the construction of the sets $[G_{a_1a_2\ldots a_n}]$ in order to insure that the continua $[axb]$ will be simple arcs from a to b. For simplicity, I shall define only the sets G_0, G_{00}, G_{000}, $\cdots$ so that the product $\bar{G}_0\cdot\bar{G}_{00}\cdot G_{000}\cdots$ will be an arc from a to b, since obviously the construction is the same for all the other sets $[G_{a_1a_2\ldots a_n}]$, it being understood that for each n, the 2^n sets $[G_{a_1a_2\ldots a_n}]$ are constructed so that if G_1 and G_2 are any two of these sets, then $\bar{G}_1\cdot\bar{G}_2=a+b$.

With the aid of the arc T_0 it is easily seen that there exists a compact simple chain R_0 of regions $V_1, V_2, V_3,\cdots V_n$ all of diameter <1 such that $\bar{V}_1\cdot T_1=a$, $\bar{V}_n\cdot T_1=b$, $\bar{V}_i\cdot T_1=0$ for $1<i<n$, $V_i\cdot V_j\neq 0$ for $i\neq j$ if and only if $|i-j|=1$, and $\bar{V}_i\cdot\overline{(R_0-V_{i-1}-V_i-V_{i+1})}=0$ for $i\leq i\leq n$, where $V_0=V_{n+1}=0$. Now, applying the lemma to the region V_1, we can obtain a compact region U_1 of diameter <1 such that $\bar{U}_1\cdot T_1=a$, $\bar{U}_1\cdot\overline{(R_0-V_1-V_2)}=0$, and such that U_1+a contains a compact continuous curve H_1 having the property that H_1-a is connected and contains V_1. There exist distinct points x and y in the set $H_1\cdot V_1\cdot V_2$. Since $H_1\supset V_1$ and M has no local separating point, it follows that no point can separate either x or y from a in H_1. Therefore † there exist arcs ax and ay in H_1 such that $ax\cdot ay=a$. On the arcs ax and ay, in the orders a, x and a, y, let x_1 and y_1 denote respectively the first points belonging to $\bar{V}_2$. Now,

* That is, x is expressed according to the dyadic number system.

† See W. L. Ayres, *loc. cit.*

applying the lemma to V_2, we can obtain a compact region U_2 of diameter < 1 containing V_2 and such that $\bar{U}_2 \cdot (T_1 + ax_1 + ay_1) = x_1 + y_1$, $\bar{U}_2 \cdot \overline{(R_0 - V_{2-1} - V_2 - V_{2+1})} = 0$ and such that $\bar{U}_2$ contains a compact continuous curve H_2 having the property that $H_2 - (x_1 + y_1)$ is connected and contains V_2. Let w and z be two points belonging to the set $H_2 \cdot V_2 \cdot V_3$. Then, just as before, no point can separate any two of the points x_1, y_1, w and z in H_2, and accordingly there exist two mutually exclusive arcs in H_2 joining the sets $x_1 + y_1$ and $w + z$. The two possible cases here are alike, so we shall suppose there exist mutually exclusive arcs x_1w and y_1z in H_2. On these two arcs, in the orders x_1, w, and y_1, z, let x_2 and y_2 respectively denote the first points belonging to $\bar{V}_3$. Then apply the lemma to V_3, and so on. Continuing this process for $n - 1$ steps, we obtain regions U_1, U_2, $\cdots, U_{n-1}$ and arcs ax_1, x_1x_2, $x_2x_3, \cdots x_{n-2}x_{n-1}$, ay_1, y_1y_2, $y_2y_3, \cdots y_{n-2}y_{n-1}$. Applying the lemma to V_n we can obtain, just as before, a region U_n and two arcs $x_{n-1}b$ and $y_{n-1}b$ satisfying similar conditions. Set $G_0 = \sum_1^n U_m$, $T_{00} = ax_1 + x_1x_2 + \cdots + x_{n-1}b$, and $T_{01} = ay_1 + y_1y_2 + \cdots + y_{n-1}b$. Then clearly G_0 is a simple chain of regions with links U_i of diameter < 1 such that $\bar{U}_1 \supset a$, $\bar{U}_n \supset b$, and T_{00} and T_{01} are arcs from a to b such that $T_{00} \cdot T_{01} = a + b$ and $x_ix_{i+1} + y_iy_{i+1} \subset U_{i+1}$ (where $0 \leq i < n$, $x_0 = y_0 = a$, $x_n = y_n = b$).

Now with the aid of the arc $T_{00} = ax_1 + \cdots + x_{n-1}b$ it is easily seen that there exists within G_0 a simple chain R_{00} of regions $Q_1, Q_2, \cdots Q_m$ all of diameter $< 1/2$ such that $\bar{Q}_1 \supset a$, $\bar{Q}_m \supset b$, etc., and such that R_{00} is the sum of n simple chains $C_1 = Q_1 + Q_2 + \cdots Q_{m_1}$, $C_2 = Q_{m_1} + Q_{m_1+1} + \cdots + Q_{m_2}, \cdots, C_n = Q_{m_{n-1}+1} + \cdots + Q_m$, where for each i, $1 \leq i \leq n$, $C_i \subset U_i$. Then by the same method as used above in the case of the chain R_0, we can define a chain G_{00} with links $S_1, S_2, \cdots S_m$ all of diameter $< 1/2$, having the same properties as stated for the chain R_{00} and in addition the property that $G_{00} + a + b$ contains two independent arcs T_{000} and T_{001} from a to b each of which is the sum of m arcs ar_1, r_1r_2, $r_2r_3, \cdots r_{m-1}b$, where for each i, $(0 \leq i < n)$, $r_ir_{i+1} \subset S_i$, where $r_0 = a$ and $r_n = b$.

Repeating this process, using T_{000}, we obtain a chain G_{000}, and so on. Continuing this process indefinitely, we obtain a sequence $G_0, G_{00}, G_{000}, \cdots$ of simple chains of regions having all the properties necessary to insure that the sets of points $axb = \bar{G}_0 \cdot \bar{G}_{00} \cdot \bar{G}_{000} \cdots$ will be a simple continuous arc from a to b. The proof that this is the case is almost identically the same as that given in a proof by R. L. Moore.*

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* See "On the Foundations of Plane Analysis Situs," *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 138.

ON THE LIBRATION POINTS OF THE RESTRICTED PROBLEM OF THREE BODIES.

By MONROE MARTIN.

INTRODUCTION.

In the restricted problem of three bodies if the constant distance between the two finite masses be taken as the unit of distance, if the unit of time be so chosen that the gravitational constant is unity, and if the unit of mass be taken as the sum of the masses of the two finite bodies, the equations of motion for the infinitesimal mass become †

$$\ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y, \tag{1}$$

where

$$\Omega(x,y) = \tfrac{1}{2}(x^2+y^2) + \frac{1-\mu}{|[(x+\mu)^2+y^2]^{\frac{1}{2}}|} + \frac{\mu}{|[(x+\mu-1)^2+y^2]^{\frac{1}{2}}|}, \qquad (0<\mu<1) \tag{2}$$

in which μ is the mass of that one of the two finite bodies (which lie on the x-axis and have the coördinates $x = 1-\mu$ and $x = -\mu$) which is situated at $x = 1-\mu$, the origin of the coördinate system being the center of mass. As is well known there exist for $0 < \mu < 1$ ‡ five and only five points of zero force called libration points whose coördinates satisfy the equation

$$\operatorname{grad}\Omega = 0, \tag{3}$$

that is

$$\begin{aligned} \Omega_x(x,y) &\equiv x - \frac{(1-\mu)(x+\mu)}{|[(x+\mu)^2+y^2]^{3/2}|} - \frac{\mu(x+\mu-1)}{|[(x+\mu-1)^2+y^2]^{3/2}|} = 0, \\ \Omega_y(x,y) &\equiv y - \frac{(1-\mu)y}{|[(x+\mu)^2+y^2]^{3/2}|} - \frac{\mu y}{|[(x+\mu-1)^2+y^2]^{3/2}|} = 0. \end{aligned} \tag{4}$$

Three of the libration points lie on the line joining the masses μ and $1-\mu$.

† Cf. for instance, T. Levi-Civita, "Sopra alcuni criteri di instabilità," *Annali di Matematico* (3), Vol. 5 (1901), pp. 282-284.

‡ For the cases where $\mu = 0$, and $\mu = 1$ there exists an infinity of points on a circle of radius unity described about the mass as a centre. That in these limit cases these points and only these are libration points follows immediately from equations (4).

These three points are separated by the masses for $0 < \mu < 1$ and their mutual position is, in the notation of E. Strömgren * as follows:

The remaining two libration points denoted by L_4 and L_5 each form an equilateral triangle with the masses μ and $1-\mu$.

In this paper the distances of L_1 and L_2 from the mass μ are designated by ρ_1 and ρ_2 respectively, while that of L_3 from the mass $1-\mu$ is designated by ρ_3. The distances ρ_1, ρ_2 and ρ_3 are functions of μ. In part I of this paper are proved the following theorems on the nature of these functions:

Theorem I. *The functions* $\rho_1(\mu)$, $\rho_2(\mu)$ *and* $\rho_3(\mu)$ *are monotone in the interval* $0 < \mu < 1$ *and have the boundary values* †

$$\text{(5a)} \qquad \begin{aligned} &\rho_1(+0)=0, \ \rho_1(1-0)=1; \\ &\rho_2(+0)=0, \ \rho_2(1-0)=1; \\ &\rho_3(+0)=1, \ \rho_3(1-0)=0, \end{aligned}$$

so that

$$\text{(5b)} \qquad 0 < \rho_k < 1 \quad \text{for} \quad 0 < \mu < 1; \qquad (k=1,2,3).$$

Moreover it is possible to obtain better inequalities for ρ_k in which the boundary values of the inequalities are functions of μ. In this connection we have

Theorem II. *The functions* $\rho_1(\mu)$, $\rho_2(\mu)$ *and* $\rho_3(\mu)$ *may be bounded as follows* ‡:

$$\begin{cases} (6a_1) & \mu < \rho_1 < \tfrac{1}{2} & \text{for } 0 < \mu < \tfrac{1}{2}, \\ (6a_2) & \rho_1 = \tfrac{1}{2} & \text{for } \mu = \tfrac{1}{2}, \\ (6a_3) & \tfrac{1}{2} < \rho_1 < \mu & \text{for } \tfrac{1}{2} < \mu < 1, \end{cases}$$

$$\begin{array}{lll} (6b) & \mu < \rho_2 < \mu^{1/4} & \text{for } 0 < \mu < 1, \\ (6c) & 1-\mu < \rho_3 < (1-\mu)^{1/4} & \text{for } 0 < \mu < 1. \end{array}$$

More definite information on the nature of the functions $\rho_1(\mu)$, $\rho_2(\mu)$ and $\rho_3(\mu)$ is given by the following theorem on the relative values of these functions for the interval $0 < \mu < 1$:

Theorem III. *There exists in the interval* $0 < \mu < 1$ *one and only one*

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* E. Strömgren, *Publikationer og mindre Meddelelser fra Köbenhavns Observatorium*, Nr. 39 (1922).

† Cf. footnote ‡ of previous page.

‡ It follows quite readily from the inequalities of Theorem II that inequalities which are functions of μ can be given for the values of $\Omega(x, y)$ at each of the libration points L_1, L_2, and L_3.

value of $\mu = \mu^*$ *for which the following relations hold between the functions* $\rho_1(\mu)$ *and* $\rho_2(\mu)$:

(7a) $\rho_1(\mu) < \rho_2(\mu)$ for $0 < \mu < \mu^*$,

(7b) $\rho_1(\mu^*) = \rho_2(\mu^*)$

(7c) $\rho_1(\mu) > \rho_2(\mu)$ for $\mu^* < \mu < 1$,

and here

(7d) $\mu^* > 1/2$.

Part II of the paper is concerned with the nature of the function $\Omega(x, y)$ at the three libration points L_1, L_2 and L_3. In his paper on the restricted problem of three bodies Birkhoff † mentions the fact that the value of $\Omega(x, y)$ at the libration point L_1 is greater than the value at either L_2 or L_3. The proof of this statement for all values of μ is not to be found in the literature. In "Die Mechanik des Himmels" by Charlier, a proof is given employing power series expansions of ρ_k, but recent numerical calculations by E. Strömgren prove that the expansions employed by Charlier are valid only for exceedingly small values of μ. Now by a simple method it is possible to prove this statement, namely (8a) below, for all values of μ in the interval $0 < \mu < 1$. In addition I shall demonstrate the statements (8b) and (8d) below which concern the relative values of $\Omega(x, y)$ at the libration points L_2 and L_3.

THEOREM IV.‡ *Denoting by* $\Omega(L_1)$, $\Omega(L_2)$ *and* $\Omega(L_3)$ *the values of the function* Ω *at the libration points* L_1, L_2 *and* L_3 *respectively,* $\Omega(L_1)$, $\Omega(L_2)$ *and* $\Omega(L_3)$ *satisfy the following relations in the interval* $0 < \mu < 1$:

(8a) $\Omega(L_1) > \Omega(L_2)$ and $\Omega(L_1) > \Omega(L_3)$ for $0 < \mu < 1$,

(8b) $\Omega(L_2) > \Omega(L_3)$ for $0 < \mu < 1/2$,

(8c) $\Omega(L_2) = \Omega(L_3)$ for $\mu = 1/2$,

(8d) $\Omega(L_2) < \Omega(L_3)$ for $1/2 < \mu < 1$.

† G. D. Birkhoff, "The Restricted Problem of Three Bodies," *Rendiconti del Circolo Matematico di Palermo*, Vol. 39 (1915), pp. 281-283.

‡ If we designate the values of the function $\Omega(x, y)$ at the points L_4 and L_5 by $\Omega(L_4)$ and $\Omega(L_5)$ respectively, we have from (2), since L_4 and L_5 lie equally distant from the x-axis, $\Omega(L_4) = \Omega(L_5)$ for $0 < \mu < 1$. The function $\Omega(x, y)$ becomes infinite at infinity and at both the masses. It has been proved by Plummer (in his paper mentioned below) that $\Omega(x, y)$ possesses a minimax at each of the libration points L_1, L_2 and L_3 and it accordingly follows that $\Omega(x, y)$ must have an absolute minimum at L_4 and L_5.

An apparent paradox arises in the distribution of the libration points L_1, L_2 and L_3 on the x-axis for the masses $\mu = 0$ and $\mu = 1$. For $\mu = 0$ the libration points L_1 and L_2 coincide and for $\mu = 1$ the libration points L_1 and L_3 coincide.* For these values of μ the actual distribution of the mass of the system is symmetrical while the libration points are placed unsymmetrically. The difficulty arises in the fact that the above distributions of the libration points are not for the distributions of mass $\mu = 0$ and $\mu = 1$, but rather for the distributions $\lim_{\mu \to 0} \mu$ and $\lim_{\mu \to 1} \mu$ which are unsymmetrical.

In the appendix the results of part I are used to prove that the function $\Omega(x, y)$ possesses a minimax at each of the libration points L_1, L_2 and L_3. This theorem was first proved by Plummer.†

PART I.

Proof of Theorem I.

For the three collinear libration points L_1, L_2 and L_3, lying on the x-axis, the second of equations (4), namely $\Omega_y(x, 0) = 0$, is identically fulfilled and the first equation will be

$$(9) \qquad \Omega_x(x, 0) = x - (1 - \mu)(x + \mu)/|x + \mu|^3 \\ - \mu(x + \mu - 1)/|x + \mu - 1|^3 = 0.$$

The real roots of this equation are the coördinates of the libration points L_1, L_2 and L_3. Now it follows from (9) that for $0 < \mu < 1$

$$(10) \quad \Omega_x(\pm\infty, 0) = \pm\infty, \quad \Omega_x(-\mu \pm 0, 0) = \mp\infty, \quad \Omega_x(1 - \mu \pm 0, 0) = \mp\infty,$$

$$(11) \qquad \Omega_{xx}(x, 0) = 1 + 2(1 - \mu)/|x + \mu|^3 + 2\mu/|x + \mu - 1|^3.$$

Therefore

$$(12) \qquad \Omega_{xx}(x, 0) > 0 \qquad \text{for} \quad 0 < \mu < 1.$$

From (10) and (12) we see that the function $\Omega_x(x, 0)$ has for $0 < \mu < 1$ one and only one zero in each of the three intervals

$$-\infty < x < -\mu, \qquad -\mu < x < 1 - \mu, \qquad 1 - \mu < x < \infty,$$

that is we have exactly three collinear libration points which furthermore are separated by the two finite masses μ and $1 - \mu$. If we denote the x-coördinates

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* Cf. Theorem I.

† H. C. Plummer, "Neighbourhood of Centers of Libration," *Monthly Notices of the Royal Astronomical Society*, Vol. 62 (1901), pp. 6-17.

of the three libration points L_1, L_2, and L_3 by x_1, x_2, and x_3 respectively, we may write

$$(13) \qquad x_1 = 1 - \mu - \rho_1, \qquad x_2 = 1 - \mu + \rho_2, \qquad x_3 = -\mu - \rho_3,$$

where ρ_1, ρ_2 and ρ_3 are the positive functions of μ defined in the introduction. Then (9) may be written in the form

$$(14a) \qquad \Omega_1(\rho_1, \mu) \equiv 1 - \rho_1 - \mu - (1-\mu)/(1-\rho_1)^2 + \mu/\rho_1^2 = 0,$$
$$(14b) \qquad \Omega_2(\rho_2, \mu) \equiv 1 + \rho_2 - \mu - (1-\mu)/(1+\rho_2)^2 - \mu/\rho_2^2 = 0,$$
$$(14c) \qquad \Omega_3(\rho_3, \mu) \equiv -\rho_3 - \mu + (1-\mu)/\rho_3^2 + \mu/(1+\rho_3)^2 = 0,$$

where (14a), (14b) and (14c) define $\Omega_1(\rho_1, \mu)$, $\Omega_2(\rho_2, \mu)$ and $\Omega_3(\rho_3, \mu)$ respectively. These equations on simplifying yield

$$(14A) \qquad \rho_1^5 - (3-\mu)\rho_1^4 + (3-2\mu)\rho_1^3 - \mu\rho_1^2 + 2\mu\rho_1 - \mu = 0,$$
$$(14B) \qquad \rho_2^5 + (3-\mu)\rho_2^4 + (3-2\mu)\rho_2^3 - \mu\rho_2^2 - 2\mu\rho_2 - \mu = 0,$$
$$(14C) \qquad \rho_3^5 + (2+\mu)\rho_3^4 + (1+2\mu)\rho_3^3 - (1-\mu)\rho_3^2 - 2(1-\mu)\rho_3 - 1 + \mu = 0.$$

The three positive functions $\rho_k(\mu)$ are defined uniquely by (14A), (14B) and (14C). We will demonstrate by reductio ad absurdum that

$$(15) \qquad \frac{d\rho_k}{d\mu} \neq 0 \qquad \text{for } 0 < \mu < 1; \qquad (k = 1, 2, 3).$$

We demonstrate (15) at first for $k = 1$. If equation (14A) be differentiated with respect to μ and the derivative of ρ_1 assumed to be zero we obtain

$$(16) \qquad \rho_1^4 - 2\rho_1^3 - \rho_1^2 + 2\rho_1 - 1 = 0.$$

Since L_1 lies between the two finite masses for all values of μ it is sufficient if we show that this quartic equation in ρ_1 has no roots in the interval $0 < \rho_1 < 1$. Denoting the left hand member of (16) by $F(\rho)$, we have

$$(17) \qquad \begin{aligned} &F(0) = F(1) = -1, && \left(\frac{dF(\rho_1)}{d\rho_1}\right)_{\rho_1 = 1/2} = F'(1/2) = 0, \\ &F(1/2) = -7/16, && F''(\rho_1) = 12\rho_1^2 - 12\rho_1 - 2. \end{aligned}$$

Since $F''(\rho_1)$ has no roots in the interval $0 < \rho_1 < 1$, we conclude that $F'(\rho_1)$ has only one root in the interval $0 < \rho_1 < \mu$, namely $\rho_1 = 1/2$, and consequently that (16) does not vanish in the interval $0 < \rho_1 < 1$. It follows the original assumption that the derivative of ρ_1 can be zero in the interval $0 < \rho_1 < 1$ is untenable.

In order to prove (15) for $k = 2$, we differentiate (14B) and obtain on assuming the derivative of ρ_2 to be zero

$$(18) \qquad \rho_2^4 + 2\rho_2^3 + \rho_2^2 + 2\rho_2 + 1 = 0.$$

Now (18) obviously has no positive roots and the proof of (15) for $k = 2$

follows immediately. It follows from considerations of symmetry that, if (15) is true for $k=2$, it is also true for $k=3$, and finally it follows from (14a), (14b) and (14c) that (5) are valid.

Proof of Theorem II.

If we place $\rho_2=\mu^{1/4}$ in (14b) we have

(19) $\Omega_2(\mu^{1/4},\mu)=1+\mu^{1/4}-\mu-(\mu^{1/2}+2\mu^{3/4}+1)/(1+\mu^{1/4})^2,$

(20) $\Omega_2(\mu^{1/4},\mu)>1+\mu^{1/4}-\mu-(\mu^{1/2}+2\mu^{1/4}+1)/(1+\mu^{1/4})^2;\ (0<\mu<1)$

and therefore

(21) $$\Omega_2(\mu^{1/4},\mu)>0 \qquad \text{for} \qquad 0<\mu<1.$$

If we now place $\rho_2=\mu$ in (14b), we obtain

(22) $$\Omega_2(\mu,\mu)=(\mu-1)(\mu^2+3\mu+1)/\mu(1+\mu^2).$$

Since $(\mu-1)(\mu^2+3\mu+1)<0$ for $0<\mu<1$, we have

(23) $$\Omega_2(\mu,\mu)<0 \qquad \text{for} \quad 0<\mu<1.$$

The inequality (6b) now follows immediately from (12), (21) and (23) and the validity of (6c) follows by symmetry from (6b).

We now demonstrate ($6a_1$), ($6a_2$) and ($6a_3$). From (14a) we have

(24) $$\Omega_1(\mu,\mu)=[1-2\mu]\,[1+\mu(1-\mu)]/\mu(1-\mu);$$

therefore

(25) $$\Omega_1(\mu,\mu)>0 \qquad \text{for} \quad 0<\mu<\tfrac{1}{2}.$$

The lower bound in ($6a_1$) follows from (12) and (25) and the upper bound in ($6a_1$) is a consequence of Theorem I. Now ($6a_2$) is obvious while ($6a_3$) follows from ($6a_1$) by symmetry.†

Proof of Theorem III.

On eliminating μ between (14A) and (14B), we have

(26) $$\frac{\rho_1^5-3\rho_1^4+3\rho_1^3}{\rho_1^4-2\rho_1^3-\rho_1^2+2\rho_1-1}+\frac{\rho_2^5+3\rho_2^4+3\rho_2^3}{\rho_2^4+2\rho_2^3+\rho_2^2+2\rho_2+1}=0.$$

We assume $\rho_1=\rho_2=\rho$ in (26) and obtain

(27) $$\rho^4(\rho^5-6\rho^3-2\rho^2+6)=0,$$

(27a) $$Q(\rho)\equiv\rho^5-6\rho^3-2\rho^2+6=0.$$

A simple calculation shows that in the interval $0<\mu<1$ the equation (27a)

† While the inequalities (6b) and (6c) are valid throughout the interval $0<\mu<1$ they give a good approximation for ρ_2 and ρ_3 only for $0<\mu<\epsilon$ and $1-\delta<\mu<1$ where ϵ and δ are small positive numbers.

has one and only one root ρ^*. From (27a) we have $Q(3/4) > 0$ and $Q(1) < 0$, that is

$$(28a) \qquad 3/4 < \rho^* < 1,$$

and by (14A)

$$(28b) \qquad 2/3 < \rho_2 < 3/4 \dagger \qquad \text{for} \quad \mu = 1/2.$$

The proof of (7d) follows from (28a) and (28b) by Theorem I.

Inequalities (7a) and (7c) will now be proven together. The proof consists in establishing that

$$(29) \qquad \frac{d\rho_1}{d\mu} \neq \frac{d\rho_2}{d\mu} \qquad \text{for} \quad \mu = \mu^*.$$

The inequalities (7a) and (7c) then follow readily from the uniqueness of μ^* inasmuch as (7a) follows from ($6a_2$), (28a) and (28b). Then (7c) is an immediate consequence of (29). To prove (29) we again appeal to a reductio ad absurdum. We assume (29) is not true, and (26) becomes, for $\mu = \mu^*$, on differentiating with respect to μ

$$(30) \qquad \rho^3(\rho^5 + 3\rho^3 + 14\rho^2 + 24) = 0.$$

But this equation has no positive roots; whence (29) follows as a necessary consequence.

Part II.

Proof of Theorem IV.

We now prove (8a). Denote by $\Omega(\rho)$ and $\Omega(-\rho)$ the values of $\Omega(x, 0)$ at points distant ρ and $-\rho$ from the mass μ respectively. Then, by (9), we have

$$(31) \qquad \begin{aligned} \Omega(-\rho) &= 1/2(1-\mu-\rho)^2 + (1+\mu)/(1-\rho) + \mu/\rho, \\ \Omega(\rho) &= 1/2(1-\mu+\rho)^2 + (1-\mu)/(1+\rho) + \mu/\rho. \end{aligned}$$

Therefore

$$(32) \qquad \Omega(-\rho) - \Omega(\rho) = 2\rho^3(1-\mu)/(1-\rho^2),$$

and accordingly

$$(33) \qquad \Omega(-\rho) - \Omega(\rho) > 0 \qquad \text{for} \quad 0 < \mu < 1; \quad 0 < \rho < 1.$$

The equality (8c) is obvious. We now give a proof of (8b) and (8d). From (31) we obtain

$$(34) \qquad \frac{\partial \Omega(\rho)}{\partial \rho} \equiv \Omega_\mu(\rho) = -(\rho^3 + (2-\mu)\rho^2 + (1-\mu)\rho - 1)/\rho(1+\rho),$$

which can be written

† While better limiting values for ρ_2 have been found, the interval given here is sufficient for the needs of this paper.

(35) $\Omega_\mu(\rho) = -(\rho^5 + (3-\mu)\rho^4 + (3-2\mu)\rho^3 - \mu\rho^2 - \rho)/\rho^2(1+\rho)^2,$

and therefore from (14B), in the notation of the introduction for Theorem IV,

(36) $$\Omega_\mu(L_2) = (\rho_2 - 2\mu\rho_2 - \mu)/\rho_2{}^2(1+\rho_2)^2.$$

Therefore from (6b)

(37) $$\Omega_\mu(L_2) < 0.$$

The inequalities (8b) and (8d) follow directly from (37).

APPENDIX.

We now show that the function $\Omega(x, y)$ has a minimax at each of the points L_1, L_2, and L_3 and therefore in L_4 and L_5 certainly an absolute minimum.† It will be sufficient to show that

(38) $$\Omega_{xx}(L_k)\Omega_{yy}(L_k) - \Omega_{xy}(L_k)^2 < 0 \qquad (k = 1, 2, 3).$$

From (2) one obtains $\Omega(x, -y) = \Omega(x, y)$. Consequently $\Omega_{xy}(x, 0) = 0$. Therefore (38) becomes, from (12),

(39) $$\Omega_{yy}(L_k) < 0 \qquad (k = 1, 2, 3).$$

From (2) and (13) we have

(40a) $$\Omega_{yy}(x_1, 0) = 1 - (1-\mu)/(1-\rho_1)^3 - \mu/\rho_1{}^3$$

and

(40b) $$\Omega_{yy}(x_2, 0) = 1 - (1-\mu)/(1+\rho_2)^3 - \mu/\rho_2{}^3.$$

As a consequence of Theorem I we have from (40a) and (13)

(41) $$\Omega_{yy}(x_1, 0) < 0 \qquad \text{for} \quad -\mu < x_1 < 1-\mu.$$

It is clear from (40b) that $\Omega_{yy}(x_2, 0)$ is negative for small values of ρ_2 and positive for sufficiently great values of ρ_2. We now show the values for ρ_2 given as a function of μ by (14B) are small enough so that for x_2 defined by (13) we have

(42) $$\Omega_{yy}(x_2, 0) < 0 \qquad \text{for} \quad 0 < \mu < 1.$$

We have from (40b)

(43) $\Omega_{yy}(x_2, 0) = (\rho_2{}^6 + 3\rho_2{}^5 + 3\rho_2{}^4 - 3\mu\rho_2{}^2 - 3\mu\rho_2 - \mu)/\rho_2{}^3(1+\rho_2)^3,$

and this reduces by (14B) to

(44) $$\Omega_{yy}(x_2, 0) = (\mu - 1)(\rho_2{}^3 + 3\rho_2 + 3)/(1+\rho_2)^3.$$

The inequality (42) follows at once from (44). The proof of (39) for $k = 1$ and 2 follows from (41) and (44) respectively; and the proof for $k = 3$ is a necessary consequence of the validity for $k = 2$.

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† Cf. foot-note ‡ to Theorem IV, p. 169.

ON ROTATIONS IN ORDINARY AND NULL SPACES.

By S. A. Schelkunoff.

1. In the following paper I am interested primarily in two problems, one of which deals with determination of invariant lines, planes and angles of rotations and quasi-rotations in a flat space of n dimensions, while the other is the converse problem.

When the paper was being written the author was not aware that either of these problems had been completely solved. He knew only of a paper * in which Professor F. N. Cole proved that every rotation in a 4-flat could be considered as a succession of two simple rotations taking place in two absolutely orthogonal planes. His method was based on direct computation in terms of Cayley's independent parameters of the coefficients of the group of rotations in a 4-flat. The method involves laborious computations even in a case of 4-flat.

At a later date, the author's attention was called to a paper written by Camille Jordan.† Jordan proved that an ordinary rotation leaves relatively invariant certain biplanes (i. e., $(n-2)$-flats immersed in an n-flat). He reached the result by concentrating his attention on infinitesimal rotations. Jordan named Schläfli as the first who had obtained equation (8) of this paper, but he claimed that the latter had not perceived its geometric significance.

The method of this paper seems to be more direct than Jordan's and it is certainly instrumental in the solution of the converse problem which appears never to have been solved in the general case of n-flat. There exists only a well-known solution for 3-flat and one for 4-flat implicitly contained in Jordan's paper.

Among the more important results obtained in this paper, equations (22) and (29) appear to be new.

Further search through literature disclosed that Ludwig Bieberbach was interested in the problem of reduction of the rotation group to a canonical

* F. N. Cole, "On Rotations in Space of Four Dimensions," *American Journal of Mathematics*, Vol. 12 (1890), pp. 191-210.

† M. Camille Jordan, "Essai sur la geométrie à n dimensions," *Bulletin de la Société Mathématique*, Vol. 3 (1875), pp. 103-174.

form and solved it on the basis of Cayley's representation of such groups in terms of independent parameters.*

Bieberbach made references to Muth,† Schläfli ‡ and Goursat § as those whose papers had touched partially on the subject.

Since our solution of the direct problem is apparently different from any other known to the author, and since we use it as a basis for the solution of the converse problem, we include it in full in the present paper.

2. The group of rotations around the origin in an ordinary n-flat is defined analytically by the following set of equations:

$$y_k = a_k^r x_r, \qquad (k, r = 1, 2, \cdots n), \quad (r \text{ umbral}) \tag{1}$$

where the coefficients a_k^r are real, subject to the conditions,

$$\begin{aligned} a_k^r a_k^s &= 1, && \text{if } r = s, \\ &= 0, && \text{if } r \neq s, \end{aligned} \tag{2}$$

and the determinant $| a_k^r |$ is equal to unity. The equivalent conditions are

$$\begin{aligned} a_r^k a_s^k &= 1, && \text{if } r = s, \\ &= 0, && \text{if } r \neq s, \end{aligned} \tag{3}$$

with the above assumption regarding the determinant.

The group of quasi-rotations is defined by similar equations:

$$y_k = A_k^r x_r, \qquad (k, r = 1, 2, \cdots n) \tag{4}$$

where the coefficients A_k^r are subject to the conditions

$$\begin{aligned} A_k^r \bar{A}_k^s &= 1, && \text{if } r = s, \\ &= 0, && \text{if } r \neq s \end{aligned} \tag{5}$$

or their equivalents,

$$\begin{aligned} A_r^k \bar{A}_s^k &= 1, && \text{if } r = s, \\ &= 0, && \text{if } r \neq s. \end{aligned} \tag{6}$$

It is easy to prove that the determinant $| A_k^r |$ is a unit complex number.

The group of quasi-rotations can be taken as a basis of "metrical" geometry in null spaces.¶

* Ludwig Bieberbach, "Uber die Bewegungsgruppen der euklidischen Räume," *Mathematische Annalen*, Vol. 70 (1911), pp. 297-336.

† Muth, *Theorie und Anwendung der Elementarteiler*, Leipzig (1899), s. 176.

‡ Schläfli, *Journal für Mathematik*, Vol. 65 (1866), s. 185.

§ Goursat, *Annales de l'École Normale Superieure* (3), t. 6 (1889).

¶ S. A. Schelkunoff, *On Certain Properties of the Metrical and Generalized Metrical Groups in Linear Spaces of n Dimensions*, Lütcke and Wulff, Hamburg, Germany, (1927).

3. If $x_1, x_2, \cdots x_n$ are the direction components of a straight line thru the origin in an n-flat (F_n) the following set of equations

$$\lambda x_k = a_k^r x_r \tag{7}$$

(where λ is the coefficient of proportionality) determines the lines of F_n invariant under transformation (1), or the axes of rotation as we might appropriately call them.

This system of linear equations has a proper solution if any only if the determinant of its coefficients vanishes, i. e., if λ is a root of the characteristic equation:

$$\begin{vmatrix} a_1^1 - \lambda & a_1^2 & a_1^3 \cdots & a_1^n \\ a_2^1 & a_2^2 - \lambda & a_2^3 \cdots & a_2^n \\ \cdots & \cdots & \cdots & \cdots \\ a_n^1 & a_n^2 & a_n^3 \cdots & a_n^n - \lambda \end{vmatrix} = 0. \tag{8}$$

The roots of equation (8), or the *rotation factors* as we might call them, *are unit complex numbers* as we can readily ascertain by multiplying equations (7) by their conjugates and then adding.

Again, if equations (7) are squared and added, we have

$$(\lambda^2 - 1) x_k x_k = 0, \tag{9}$$

i. e. either $\lambda = \pm 1$, or the axis is a null line.

If $\lambda = 1$, we have an absolutely invariant axis, i. e. a line of invariant points.

Since the coefficients of (8) are real, the complex rotation factors are grouped in conjugate pairs. Thus, null axes exist in conjugate pairs. Each such pair determines a real invariant plane that rotates on itself, as we shall prove later.

Equation (8) can be written in the form

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - S_3\lambda^{n-3} + \cdots + (-1)^{n-1}S_{n-1}\lambda + (-1)^n = 0,$$

where S_{n-k} is the sum of the principal minors of $(n-k)$-th order, taken without repetition.

Since the complementary minors of an orthogonant are equal and

$$S_k = S_{n-k},$$

this equation is symmetric if n is even, and antisymmetric if n is odd.

Hence, if n is odd, $+1$ is a root of (8), that is, *in space of odd number of dimensions there is always an absolutely invariant line.**

* According to Jordan this result was first discovered by Schläfli.

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If n is even, equation (8) can be transformed into equation of lower degree by the following substitution:

$$\lambda + 1/\lambda = 2z.$$

If n is odd, the above substitution can be used after equation (8) has been divided by $(\lambda - 1)$.

Therefore, $+1$ *is a multiple root of odd or even order if the space has respectively odd or even number of dimensions;* -1 *is always a multiple root of even order.* (Here an absence of a root is regarded as even multiplicity of zero order).

If we multiply equations (7) corresponding to λ_1 and λ_2, and simplify the result we have:

(10) $$(\lambda_1\lambda_2 - 1)x_k{}^1 x_k{}^2 = 0,$$

i. e., either $$x_k{}^1 x_k{}^2 = 0,$$

or, $$\lambda_2 = 1/\lambda_1 = \bar{\lambda}_1.$$

Hence, *any two non-conjugate axes are orthogonal.*

Multiplying equations (7) corresponding to λ_1 and $\bar{\lambda}_2$ we have:

$$(\lambda_1\bar{\lambda}_2 - 1)x_k{}^1 \bar{x}_k{}^2 = 0,$$

i. e., either

(10_1) $$x_k{}^1 \bar{x}_k{}^2 = 0,$$

or, $$\lambda_2 = \lambda_1.$$

Hence, *any two axes not having the same rotation factor are quasi-orthogonal.*

Obviously, *if for a given λ equations* (7) *have exactly "m" linearly independent solutions, this λ must be a multiple root of at least m-th order.*

From the canonical form given by Bieberbach * it follows at once that *for a given λ equations* (7) *have m linearly independent solutions if and only if λ is a multiple root of m-th order.*

4. *A conjugate pair of axes determines a real invariant plane. This plane rotates on itself.* In fact, if $\| u_r \|$ and $\| \bar{u}_r \|$ are the direction components of axes whose rotation factors are $e^{\pm i\phi}$, we can take $(u_k + \bar{u}_k)/2^{1/2}$ and $(u_k - \bar{u}_k)/i2^{1/2}$ as the direction cosines of two real straight lines in the plane determined by the axes provided $\| u_r \|$ is a quasi-normalized set, i. e.

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* *Ibid.*, p. 302.

$$u_r\bar{u}_r = 1.$$

Now if

(11) $$x_k = (u_k + \bar{u}_k)/2^{1/2}$$

are the direction cosines of a line before the transformation (1), the new direction cosines are given by:

$$y_k = a_r x_r = (\lambda u_k + \lambda\bar{u}_k)/2^{1/2},$$

and the angle thru which this line is rotated is determined from the following equation:

(12) $$\cos\theta = x_k y_k = (\lambda + \bar{\lambda})/2 = \cos\phi,$$

i. e., $$\theta = \pm\phi.$$

Similarly we can show that the line whose direction cosines are $(u_k - \bar{u}_k)/i2^{1/2}$, and, later, that every other line in the plane of rotation rotates thru the same angle ϕ.

The invariant planes corresponding to different conjugate sets of rotation factors are absolutely orthogonal. Indeed, if

$$x_k = au_k + b\bar{u}_k$$

are the direction cosines of a line thru the origin in the plane determined by one conjugate set of axes, and

$$y_k = cv_k + d\bar{v}_k$$

is a similar line in another plane of rotation, we have:

$$\begin{aligned} x_k y_k &= (au_k + b\bar{u}_k)(cv_k + d\bar{v}_k) \\ &= (acu_k v_k + bd\bar{u}_k\bar{v}_k + adu_k\bar{v}_k + bc\bar{u}_k v_k) = 0, \end{aligned}$$

provided the sets of rotation factors are distinct. This proves the above theorem.

Thus, *if the rotation factors are all different a rotation in F_{2n} can be uniquely decomposed into "n" simple rotations taking place in a set of "n" absolutely orthogonal real planes, thru angles determined by the roots of λ-equation.*

Also, *if the rotation factors are all different, a rotation in F_{2n+1} leaves absolutely invariant one real line, and relatively invariant a unique set of absolutely orthogonal (mutually as well as to the invariant line) real planes. The lines of any one invariant plane rotate thru the same angle.**)

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* These two theorems are most interesting special cases of the obvious general theorem: *In every flat space F_n there are Subflats (of 1, 2, 3, . . . n — 1 dimensions)*

Besides real invariant planes there are imaginary. In F_{2n} all of these $[2n(n-1)$ in number$]$ are null planes, as may be readily ascertained; while in F_{2n+1}, besides $2n(n-1)$ null planes, there are also $2n$ aeolotropic invariant planes, namely those determined by a null axis and the absolutely invariant real axis.

5. Interesting exceptions arise when the characteristic equation has multiple roots. As we have already stated equations (7) possess m linearly independent solutions, if the corresponding λ is a multiple root of order m. Thus, to every such root there corresponds an m-fold pencil of invariant lines.

If $\lambda = 1$ we have an absolutely invariant m-subflat.

If $\lambda = -1$ (which corresponds to rotation thru 180°), we have an invariant m-subflat every line of which turns thru 180°.

If λ is a *bona fide* complex number, $\bar{\lambda}$ is also a multiple root of order m, and hence, there are two conjugate m-fold pencils of invariant lines. Each conjugate set of lines determines a real plane. Thus, instead of the usual m real planes of rotation we have ∞^{m-1} such planes, all of which turn upon themselves and thru the same angle equal to $\cos^{-1}(\bar{\lambda} + \lambda)/2$.

6. Some of the results that we have just obtained can be readily extended to the quasi-orthogonal group defined by equations (4).

As before, the invariant lines (axes of quasi-rotations) are determined by the set of equations,

$$\lambda x_k = A_k{}^r x_r, \tag{13}$$

which possess proper solutions if any only if λ is a root of the characteristic equation:

$$\begin{vmatrix} A_1{}^1 - \lambda & A_1{}^2 & \cdots & A_1{}^n \\ A_2{}^1 & A_2{}^2 - \lambda & \cdots & A_2{}^n \\ \cdots & \cdots & \cdots & \cdots \\ A_n{}^1 & A_n{}^2 & \cdots & A_n{}^n - \lambda \end{vmatrix} = 0. \tag{14}$$

Unless this equation has multiple roots there are n and only n axes of rotation.

Again it is easy to demonstrate that

$$\lambda\bar{\lambda} = 1, \tag{15}$$

i. e., that the rotation factors are unit complex numbers.

If x_r are the direction "quasi-cosines" of an axis, i. e., if

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determined by the corresponding number of invariant lines (7), *all points of which are permuted among themselves when transformation* (1) *is applied.*

$$x_r\bar{x}_r = 1,$$

the equations

$$y_k = A_k{}^r x_r,$$

determine the new direction quasi-cosines and the "twist" of the axis is given by

$$(16) \qquad \cos\Phi = (x_k\bar{y}_k + \bar{x}_k y_k)/2 = (\lambda x_k\bar{x}_k + \bar{\lambda} x_k\bar{x}_k)/2 = (\lambda + \bar{\lambda})/2.$$

Any two axes are, in general, quasi-orthogonal. Indeed if,

$$(17) \qquad \lambda_1 x_k{}^1 = A_k{}^r x_r \quad \text{and} \quad \lambda_2 x_k{}^2 = A_k{}^s x_s,$$

determine a pair of axes, we have

$$\lambda_1\bar{\lambda}_2 x_k{}^1\bar{x}_k{}^2 = A_k{}^r\bar{A}_k{}^s x_r{}^1\bar{x}_s{}^2,$$

or,

$$(18) \qquad (\lambda_1\bar{\lambda}_2 - 1)x_k{}^1\bar{x}_k{}^2 = 0.$$

Therefore,

$$x_k{}^1\bar{x}_k{}^2 = 0,$$

unless $\lambda_2 = \lambda_1$, i. e., unless the rotation factors of both axes are the same. If the latter is the case, the axes may or may not be quasi-orthogonal, and there are more than minimum number of axes.

7. Suppose we have a set of axes whose direction quasi-cosines form the following quasi-orthogonal matrix:

$$(19) \qquad \left\Vert \begin{matrix} x_1{}^1 & x_2{}^1 & \cdots & x_n{}^1 \\ x_1{}^2 & x_2{}^2 & \cdots & x_n{}^2 \\ \cdot & \cdot & \cdot & \cdot \\ x_1{}^n & x_2{}^n & \cdots & x_n{}^n \end{matrix} \right\Vert$$

and let the corresponding quasi-rotation factors be $\lambda_1, \lambda_2, \cdots \lambda_n$. Let us determine the coefficients of the corresponding quasi-orthogonal transformation.

We have the following system of equations at our disposal:

$$(20) \qquad \begin{aligned} \lambda_1 x_k{}^1 &= A_k{}^r x_r{}^1, \\ \lambda_2 x_k{}^2 &= A_k{}^r x_r{}^2, \\ &\cdots\cdots \\ \lambda_n x_k{}^n &= A_k{}^r x_r{}^n. \end{aligned} \qquad (k, r = 1, 2, \cdots n).$$

Fixing the value of k we obtain a system of n equations with n unknowns

$A_k^1, A_k^2, \cdots A_k^n$. There are n such systems corresponding to different values of k. Solving these systems, we have:

$$A_s^k = \lambda_r x_s^r \bar{x}_k^r, \qquad (k, s = 1, 2, \cdots n), \quad (r = 1, 2, \cdots n(\text{umbral})). \tag{21}$$

We shall prove now that A_s^k given by equations (21) are actually the coefficients of a quasi-orthogonal group. Indeed, take

$$A_s^m = \lambda_p x_s^p \bar{x}_m^p. \tag{22}$$

Multiplying equations (21) by the conjugates of (22) and summing, we have

$$A_s^k \bar{A}_s^m = \lambda_r \bar{\lambda}_p x_s^r \bar{x}_k^r \bar{x}_s^p x_m^p. \tag{23}$$

But since

$$x_s^r \bar{x}_s^p = 1, \quad \text{if } p = r,$$
$$= 0, \quad \text{if } p \neq r,$$

equations (23) take the following form:

$$A_s^k \bar{A}_s^m = \lambda_r \bar{\lambda}_r x_m^r \bar{x}_k^r = x_m^r \bar{x}_k^r,$$

i. e.,

$$A_s^k \bar{A}_s^m = 1, \quad \text{if } k = m, \tag{24}$$
$$= 0, \quad \text{if } k \neq m,$$

which proves that A_s^k *given by equations* (21) *are actually the coefficients of a quasi-orthogonal group that leaves invariant a set of lines determined by matrix* (19).

The ordinary orthogonal group is obtained if the invariant axes given by matrix (19) *are conjugate in pairs and the corresponding rotation factors are also conjugate. If any of the axes happens to be real, the corresponding* λ *must be either* $+1$ *or* -1 (-1 *is allowable only if there is even number of real axes*). Under such conditions, we have real A_s^k, i. e.,

$$A_s^k = \bar{A}_s^k. \tag{25}$$

In fact, equations (25) are equivalent to

$$\lambda_1 x_s^1 \bar{x}_k^1 + \lambda_2 x_s^2 \bar{x}_k^2 + \cdots = \bar{\lambda}_1 \bar{x}_s^1 x_k^1 + \bar{\lambda}_2 \bar{x}_s^2 x_k^2 + \cdots,$$

which is an obvious identity if the above conditions hold.

From the results of section 3 we conclude that the above conditions are necessary with the exception of the quasi-orthogonality condition which must be satisfied only for distinct rotation factors. However, in the latter case

there are infinitely many axes, out of which quasi-orthogonal pairs can be chosen.

Thus, equations (21) may serve for determinations of the coefficients of the group of rotations if the axes and the rotation factors are known.

In practice, however, it is more convenient to describe the rotation group in terms of known planes and angles of rotation.

8. Therefore, assume the equations of known planes of rotation in the following form:

$$x_k^m = p_k^m u + q_k^m v, \tag{26}$$

where the superscript refers to the planes while the subscript, as usual, to the coördinates of a point in the plane. Without loss of generality we may assume p_k^m and q_k^m to be the direction cosines of pairs of orthogonal lines in the corresponding planes.

Since the rotation axes are the null lines in planes (26), we can determine them immediately from the condition

$$x_k^m x_k^m = 0, \qquad \text{(only } k \text{ is umbral).} \tag{27}$$

In fact, we have

$$v = \pm iu,$$

and, hence, the direction quasi-cosines of the axes are given by:

$$x_k^m = (p_k^m \pm iq_k^m)/2^{1/2}. \tag{28}$$

Assume the corresponding rotation factors $e^{\pm i\phi_m}$. Substituting in equations (21) we have

$$a_s^k = 1/2 \sum_m [(p_s^m + iq_s^m)(p_k^m - iq_k^m)e^{i\phi_m} + (p_s^m - iq_s^m)(p_k^m + iq_k^m)e^{-i\phi_m}],$$

and simplifying,

$$a_s^k = \sum_m [(p_s^m p_k^m + q_s^m q_k^m)\cos\phi_m + (p_s^m q_k^m - p_k^m q_s^m)\sin\phi_m]. \tag{29}$$

These equations remain true for $s = k$ provided we waive the summation convention with regard to these letters. It is interesting to note that the first group of terms is symmetric in s and k and an even function of each ϕ_m, while the second group is antisymmetric in s and k and an odd function of each ϕ_m. In case of an odd space in which there exists an invariant real line the formula (29) still holds if we merely take the direction cosines of the line as p's and let q's and the corresponding ϕ be zero. We observe that

the coefficients of rotations in absolutely orthogonal planes or higher manifolds are additive. This fact may be successfully used in building up rotation groups in higher spaces when planes and angles of rotation are known.

9. It is interesting to apply formula (29) to special cases. In three dimensional space it is convenient to begin with the matrix:

$$\begin{Vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{Vmatrix}$$

the first row of which is made up of the direction cosines of the real axis of rotation, and the remaining two rows are the direction cosines of two perpendicular lines in the plane of rotation. Let the angle of rotation be ϕ. Applying equations (29) and eliminating $\beta_1, \beta_2, \cdots \gamma_3$ by means of properties of orthogonal matrices, we obtain the following equations of the group of rotations in F_3:

$$\begin{aligned} y_1 &= (2a_1^2 + 2d^2 - 1)x_1 + 2(a_1a_2 + a_3d)x_2 + 2(a_1a_3 - a_2d)x_3, \\ (30) \quad y_2 &= 2(a_1a_2 - a_3d)x_1 + (2a_2^2 + 2d^2 - 1)x_2 + 2(a_2a_3 + a_1d)x_3, \\ y_3 &= 2(a_1a_3 + a_2d)x_1 + 2(a_2a_3 - a_1d)x_2 + (2a_3^2 + 2d^2 - 1)x_3, \end{aligned}$$

where,

$$d = \cos(\phi/2), \qquad a_k = \cos\alpha_k \sin(\phi/2).$$

These formulae are identical with those given by L. E. Dickson,* except for the convention concerning the direction of rotation.

In four-dimensional space we may start with a plane

$$(31) \qquad x_3 = ax_1 + bx_2, \qquad x_4 = cx_1 + dx_2,$$

that rotates on itself thru angle ϕ. Let the plane absolutely orthogonal to (31) rotate thru angle ψ.

The equation of the plane absolutely orthogonal to the plane (31) may be written

$$x_1 = -ax_3 - cx_4, \qquad x_2 = -bx_3 - dx_4.$$

The plane (31) contains the points $(1, 0, a, c)$ and $(0, 1, b, d)$. The corresponding set of p's and q's are the direction cosines of the bisectors of the angles between the lines determined by the origin and those points. Similarly for the other plane.

If we use the following abbreviations

* L. E. Dickson, *Modern Algebraic Theories*, p. 100.

$$D = ad - bc, \quad R^2 = 1 + a^2 + b^2 + c^2 + d^2 + D^2,$$
$$A^2 = 1 + a^2 + c^2, \qquad A^{1^2} = R^2 - A^2,$$
$$B^2 = 1 + b^2 + d^2, \qquad B^{1^2} = R^2 - B^2,$$
$$P^2 = 1 + a^2 + b^2, \qquad P^{1^2} = R^2 - P^2,$$
$$Q^2 = 1 + c^2 + d^2, \qquad Q^{1^2} = R^2 - Q^2,$$

we can write the coefficients of the group of rotations as follows:

$$\begin{aligned}
R^2a_1{}^1 &= B^2 \cos\phi + B^{1^2} \cos\psi\\
R^2a_2{}^2 &= A^2 \cos\phi + A^{1^2} \cos\psi\\
R^2a_3{}^3 &= Q^{1^2} \cos\phi + Q^2 \cos\psi\\
R^2a_4{}^4 &= P^{1^2} \cos\phi + P^2 \cos\psi,
\end{aligned}$$

$$\begin{aligned}
(32) \qquad R^2a_1{}^2 &= (ab + cd)(-\cos\phi + \cos\psi) - R(\sin\phi + D\sin\psi),\\
R^2a_1{}^3 &= (a + dD)(\cos\phi - \cos\psi) - R(b\sin\phi + c\sin\psi),\\
R^2a_1{}^4 &= (c - bD)(\cos\phi - \cos\psi) - R(d\sin\phi - a\sin\psi),\\
R^2a_2{}^3 &= (b - cD)(\cos\phi - \cos\psi) - R(-a\sin\phi + d\sin\psi),\\
R^2a_2{}^4 &= (d + aD)(\cos\phi - \cos\psi) + R(c\sin\phi + b\sin\psi),\\
R^2a_3{}^4 &= (ac + bd)(\cos\phi - \cos\psi) - R(D\sin\phi + \sin\psi),
\end{aligned}$$

and $a_r{}^s$ is obtained from $a_s{}^r$ by changing simultaneously the signs of ϕ and ψ.

If $\psi = \phi$, equations (32) degenerate into

$$(33) \quad \begin{aligned}
&a_1{}^1 = a_2{}^2 = a_3{}^3 = a_4{}^4 = \cos\phi, && Ra_1{}^2 = Ra_3{}^4 = -(1 + D)\sin\phi,\\
&Ra_1{}^3 = -Ra_2{}^4 = -(b + c)\sin\phi, && Ra_1{}^4 = Ra_2{}^3 = (a - d)\sin\phi.
\end{aligned}$$

Since,

$$R^2 = (a - d)^2 + (b + c)^2 + (1 + D)^2,$$

coefficients $a_s{}^r$ depend only on two parameters besides the angle of rotation. Hence one parameter in (31) is arbitrary, and there exist ∞^1 *real* planes thru the origin that rotate on themselves, which is in keeping with the previously stated general theorem (end of section 5).

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GENERATING INVOLUTIONS OF INFINITE DISCONTINUOUS CREMONA GROUPS OF S_4 WHICH LEAVE A GENERAL CUBIC VARIETY INVARIANT.

By VIRGIL SNYDER AND MARGUERITE LEHR.

The present paper derives the equations and obtains the table of characteristics of three types A, B, C of involutorial birational transformations of S_4; they are obtained by means of the general cubic variety which remains invariant.

Of these three types the first and second each contain six parameters, and the third more. The product of any two transformations of the set, of the same or of different types, is a non-periodic transformation.

These types and two others discussed previously * generate all known transformations which leave the variety V_3 invariant.

A. Given a general cubic variety V_3 of four way space S_4 and a line l not lying on it. A birational transformation of the S_4 in which V_3 is immersed may be constructed as follows. Let M be one of the points in which l meets V_3. A point P determines the plane P, l which meets V_3 in a plane cubic curve. The tangent to it at M meets it in the tangential M'. The line $M'P$ meets the polar quadric variety of M' as to V_3 in M''. The harmonic conjugate of P as to $M'M''$ is P'. The transformation $P \sim P'$ is birational and involutorial, and under it V_3 remains invariant. Call it I.

The locus of M' is the cubic surface γ_3, intersection of V_3 with the tangent space at M; it has a node at M, hence every line of the bundle M in the tangent space τ meets it in just one point M'. The plane M', l cuts from the first polar of M' as to V_3 a conic c_2; the tangent space at M' meets l in one point, and the line joining this point to M' is the tangent line t to c_2 at M'.

In the plane l, M' the transformation I is a perspective quadratic involution determined by c_2 and M'. The only fundamental point is M', and its associated principal locus is the tangent t which meets l. The conjugates of the lines of the plane under I are conics having t for common tangent at M' and having three point contact with each other.

As the plane l, M' takes all positions about l, the locus of t is a fundamental ruled variety Γ conjugate of γ_3. The locus of c_2 is the variety H of invariant points. It touches Γ at every point of γ_3.

* *Rendiconti Circolo Matematico di Palermo*, Vol. 38 (1914), pp. 344-352.

Every plane through l is transformed into itself. In each position, l has an image conic having three point contact with the image conics of the other lines of the plane at M'. The locus of this conic, as M' describes γ_3, is the complete conjugate L of l.

2. When the section of the polar quadric at M' with the plane l, M' is composite, and the cubic section is not, one component is the tangent t and the other, the harmonic polar of the point of inflexion M' of the plane section V_3, l, M', is the axis of a harmonic homology having M' for center. Thus in each such plane the transformation is linear; the conjugate of any line in the plane is made quadratic by adjoining the inflexional tangent t. Hence every inflexional tangent of V_3 at points of γ_3, which meets l is a fundamental line of the second kind; the image of any point on one of these lines is the whole line t passing through it. These lines are therefore on H, on Γ, on L, and on the conjugate of every S_3 of S_4.

Since the conjugates of the $| S_3 |$ are varieties having three point contact on γ_3, its conjugate Γ appears three times as component of the Jacobian of these conjugate varieties. The other component is L, the conjugate of l.

3. A general S_3 meets its own conjugate in the section made by it on H and in the elliptic cubic cone with vertex on l and containing the plane cubic curve S_3, γ_3. Every line meeting l and γ_3 contains an infinite number of pairs of conjugate points, hence the involution I belongs to the generalized monoidal type, analogous to those thus defined in S_3 by Montesano.

4. Analytic procedure. Let $x_1=0$, $x_2=0$, $x_3=0$ be l and let M be $(0, 0, 0, 0, 1)$. Let $x_4=0$ be the tangent space to V at M. Then the equation of V has the form

$$V \equiv x_4 x_5^2 + x_5 f + \phi = 0,$$

in which f is a quadratic and ϕ a cubic quaternary form in x_1, x_2, x_3, x_4. The equations of γ_3 are then

$$\gamma_3 : x_4 = 0, \qquad x_5 f_0 + \phi_0 = 0,$$

f_0, ϕ_0 being f, ϕ with x_4 replaced by zero.
A point (y) determines the plane

$$x_1/y_1 = x_2/y_2 = x_3/y_3,$$

through l. This plane meets γ_3 in M' defined by

$$M' : f_0 y_1,\ f_0 y_2,\ f_0 y_3,\ 0,\ -\phi_0.$$

The polar quadric of M' as to V is

$$f_0 y_1 (x_5 f_1 + \phi_1) + f_0 y_2 (x_5 f_2 + \phi_2) + f_0 y_3 (x_5 f_3 + \phi_3) - \phi_0 (2 x_4 x_5 + f) = 0,$$

in which f_i, ϕ_i are partial derivatives of f, ϕ as to x_i. By eliminating y_1, y_2, y_3 between this equation and those of the plane l, (y) we obtain the equation of H, the locus of invariant points,

$$H: f_0 x_1(x_5 f_1 + \phi_1) + f_0 x_2(x_5 f_2 + \phi_2) + f_0 x_3(x_5 f_3 + \phi_3) - \phi_0(2x_4 x_5 + f) = 0.$$

It is of order 5, contains l to multiplicity 3, and contains γ_3.

The tangent S_3 to V_3 at M' is

$$\begin{aligned} &x_1(-\phi_0 f_0 f_{1,0} + f_0^2\phi_{1,0}) + x_2(-\phi_0 f_0 f_{2,0} + f_0^2\phi_{2,0}) \\ &\quad + x_3(-\phi_0 f_0 f_{3,0} + f_0^2\phi_{3,0}) + x_4(\phi_0^2 - \phi_0 f_{4,0} f_0 + f_0^2\phi_{4,0}) + x_5 \cdot f_0^3 = 0. \end{aligned}$$

By eliminating y as before, this equation represents Γ. It is of order 7, contains l to multiplicity 6, and contains γ_3. Any plane through l contains one and only one generator of Γ. Any point on the line $M'y$ has coördinates of the form

$$x_1 = \sigma f_0 y_1 + \tau y_1, \quad \text{etc.}$$

The point M'' in which this line meets the polar quadric of M' again is given by $\tau^2 H(y) + 2\sigma\tau H(y, M') = 0$ or

$$\tau = 2H(y, M'), \qquad \sigma = -H(y),$$

where $H(y, M') = 0$ is the polar of M' as to $H = 0$. The harmonic conjugate of y as to M', M'' is then the point

$$\begin{gathered} \rho y_1' = (\Gamma - Hf_0)y_1, \quad \rho y_2' = (\Gamma - Hf_0)y_2, \quad \rho y_3' = (\Gamma - Hf_0)y_3, \\ \rho y_4' = \Gamma y_4, \quad \rho y_5' = \Gamma y_5 + H\phi_0. \end{gathered}$$

Thus, the transformation of order 8 has

$$L: \Gamma - Hf_0 = 0$$

for the image of l. The Jacobian is of order $5(n-1) = 35$. It consists of L to multiplicity two and Γ to multiplicity three.

$$J: \Gamma^3(\Gamma - Hf_0)^2 = 0.$$

A table of characteristics of I_8 may now be constructed. Everything is determined except the loci of parasitic lines, or fundamental lines of the second kind.

5. The six planes π, defined by $f_0 = 0$, $\phi_0 = 0$ are double on each V_8, conjugates of the $| S_3 |$. These planes are fundamental of the second kind, that is, the whole plane is the conjugate of any point on it. From the forms of the equations it follows that these planes are double on L_7, Γ_7 and simple on H. The line l is six fold on each V_8; every point P of l has for conjugate

a surface of order 6, π_6 lying on the three dimensional cone P, γ_3. The tangents to V_8 at points of γ_3 which meet l form the ruled variety Γ; an arbitrary S_3 meets it in a surface of order 7, having a five fold point at the point S_3, l. No plane section c_3 of γ_3 can lie in a plane meeting l except at M, as l does not lie in the tangent space τ containing γ_3. An arbitrary S_3 meets γ_3 in a plane section c_3, and meets l in a point. The lines of Γ from points of c_3 do not lie in this S_3. They form a ruled surface of S_4, of order 9.

The other fundamental lines of the involution are the inflexional tangents to V_3 at points γ_3 which meet l. These lines form a ruled surface F_{17} of S_4, of order 17, having l for six fold line, as can be seen by passing an S_3 through l. It meets γ_3 in a plane cubic curve g_3 having a node at M. The ruled surface of inflexional tangents to $G \equiv V, S_3(l)$ has g_3 for double curve and its plane has three inflexional and two nodal tangents at M. Hence the surface is of order 11. The same result can be obtained by expressing the condition that the plane l, M' shall touch the polar quadric of M' as to G. Thus in any S_3 through l are 11 inflexional tangents to V_3 at points of γ_3 which meet l. The order of the surface in S_4 is therefore 11 plus the multiplicity of l upon it. From any point of l can be drawn six such tangents, hence the order of the surface is 17. The conjugate of any point on every generator is the entire generator passing through it. The table of characteristics can now be constructed as follows:

$$
\begin{aligned}
S_3 &\sim V_8 : 6\pi^2 l^6 \gamma_3{}^{(3)} F_{17} \\
S_2 &\sim M_{14}{}^2; \\
S_1 &\sim C_8;\ [C_8; S_1] = 5 \\
l &\sim L_7 : 6\pi^2 l^5 \gamma_3{}^{(3)} F_{17} \\
\gamma_3 &\sim \Gamma_7 : 6\pi^2 l^6 \gamma_3{}^{(2)} F_{17} \\
H_5 &\sim H_5 : 6\pi l^3 \gamma_3{}^{(2)} F_{17}.
\end{aligned}
$$

The symbol $\gamma_3{}^{(3)}$ means that all the V_8 of the system have three point contact with each other at every point of γ_3. The intersection of V_8 and L_7, a composite surface of order 56, consists of $6 \times 2 \times 2 = 24$ for the double planes, γ_3 counted three times, F_{17} and the surface of order 6, conjugate of the point in which the conjugate S_3 of the given V_8 meets l. Similarly, the intersection of V_8 and Γ_7 consists of $6\pi^2 = 24$, F_{17}, γ_3 taken twice, and the ruled surface of tangents to V_8 at points of S, γ_3 which meet l. This surface is of order 9.

The conjugate of an arbitrary plane is a surface of order 14 containing the quintic curve in which the given plane meets H. When the given plane meets l, its conjugate consists of a surface of order 8 and the sextic surface, image of the point on l.

The conjugate of a line is a curve of order 8 meeting it in 5 points. If the line meets l, its proper image is a conic having three point contact with γ_3.

6. Let l meet V in a second point N. If $N \equiv (0, 0, 0, 1, 0)$ and the tangent space to V at N be taken as $x_5 = 0$, the equation of V has the form

$$V: x_4{}^2 x_5 + x_4 x_5{}^2 + a x_4 x_5 + b x_4 + b' x_5 + c = 0$$

wherein a, b, b', c are ternary, of orders 1, 2, 2, 3, respectively, or

$$V: x_4{}^2 x_5 + x_4 g + \psi = 0.$$

If ψ^0, g^0 denote the values of ψ, g when $x_5 = 0$ then $\psi^0 = \phi_0$. The equations of the transformation S associated with N are of the same form as those of T, and can be obtained from them by making a few obvious changes.

In TS, $L_7{}^6$ appears as a factor in the second members of the equations, hence the transformation is of order 22, and is not periodic. The 12 parasitic planes all lie on the cubic cone $\phi_0 = 0$ which has the line l for vertex.

The line l meets V in a third point P: $(0, 0, 0, 1, -1)$, with $a + x_4 + x_5 = 0$ for tangent space. Let the transformation associated with this point be denoted by U. The three involutions T, S, U generate a noncyclic infinite discontinuous group.

7. Let l_1, l_2 be two lines through M, and T_1, T_2 the corresponding transformations. Their product is not periodic. The plane of $l_1 l_2$ is composed of invariant points under $T_1 T_2$.

The general transformation of this type may be thought of as follows: Let F be any rational surface of order n lying on V_3, and l any line not on V_3, meeting F in $n - 1$ points. Then any plane through l meets F in one residual point, which takes the place of M'. But $n \not> 4$, since l is not on V_3. Since V_3 contains no surfaces other than complete intersections, it follows that the surfaces γ_3 in tangent S_3 are the only possible surfaces satisfying the conditions.

8. B. Given two skew lines l_1, l_2 on V_3 and a plane π not on V_3. Given any point (y) on S_4. The $S_3 \equiv \pi$, (y) meets l_i in P_i. The line $P_1 P_2$ meets V_3 in K. The line (y), K meets the polar quadric of K as to V_3 in K'. The harmonic conjugate (y') of (y) as to K, K' determines an involutorial birational transformation I under which V_3 remains invariant.

The two lines l_1, l_2 determine an S_3 meeting V_3 in a cubic surface F on which K lies. Since P_1, P_2 are projective, the line $P_1 P_2$ describes a quadric surface meeting F in a residual rational C_4, having three points on each line l_i. This C_4 is the locus of K. Every line $P_1 P_2$ lies in some S_3 through π,

hence the line meets π. The S_3 of l_1, l_2 therefore meets π in a directrix of the quadric, that is, π meets C_4 in 3 collinear points.

Let π be defined by $x_1 = 0$, $x_2 = 0$ and C_4 by

$$x_1 = \lambda_1\phi(\lambda_1, \lambda_2), \quad x_2 = \lambda_2\phi, \quad x_3 = f_3(\lambda_1, \lambda_2), \quad x_4 = f_4(\lambda_1, \lambda_2), \quad x_5 = 0$$

wherein ϕ is a cubic and each f_i a quartic form in (λ). The polar quadric of a point (λ) on C_4 as to V is

$$H(x) : \lambda_1\phi(\lambda_1, \lambda_2)\partial V/\partial x_1 + \lambda_2\phi\partial V/\partial x_2 + f_3\partial V/\partial x_3 + f_4\partial V/\partial x_4 = 0.$$

The space $x_1y_2 - x_2y_1 = 0$ or π, (y) meets C_4 in $\lambda_1y_2 - \lambda_2y_1 = 0$, $\lambda_1 = y_1$, $\lambda_2 = y_2$. The line $(y)(\lambda)$ meets $H(x)$ in (λ) and in the point $2H(y, \lambda)(y) = H(y)(\lambda)$. The point (y') is given by

$$I_{10}: \begin{array}{ll} y_1' = [H(y, \lambda) - H(y)\phi(\lambda)]\, y_1, & y_2' = [H(y, \lambda) - H(y)\phi(\lambda)]\, y_2, \\ y_3' = H(y, \lambda)y_3 - H(y)f_3(\lambda), & y_4' = H(y, \lambda)y_4 - H(y)f_4(\lambda), \\ \multicolumn{2}{c}{y_5' = H(y, \lambda)y_5,} \end{array}$$

in which $H(y, \lambda) = 0$ is the polar of the point (λ) on C_4 as to $H = 0$. The transformation is of order 10. The plane π is of multiplicity 8 and C_4 simple on the $| V_{10} |$, conjugates of $| S_3 |$.

9. The tangent S_3 to V_3 at a point (λ) on C_4 meets π in a line. The pencil of tangents at (λ) meeting this line are all generators of $H(y, \lambda)$. The same tangent S_3 meets the polar quadric variety of (λ) in a quadric cone. It meets π in a line having two points on the cone. Lines thru these points meet V_3 in three points coincident at (λ), hence these lines are parasitic. They generate a ruled surface containing C_4 doubly. The section of the space π, (y) with the polar quadric variety is a quadric surface lying on $H(y)$. When this quadric surface is a cone, it lies on $H(y)$ and $H(y, \lambda)$; each generator is a parasitic line. It is part of the base of $| V_{10} |$, and lies on Π_9, the conjugate of π in I_{10}. The cone is the residual intersection of $H(y)$ and the tangent space π, (y) to V_3.

The plane π meets V_3 in a general cubic curve γ_3. One line in π meets γ_3 in three points on C_4; another, not in π, meets it in points of l_1, l_2, C_4. The image of any point in π is a curve of order 8, having the point to order 4, and lying on the cone having the given point as vertex, with C_4 as directrix curve. Any line meeting π in one point has for image a conic meeting it in two points. Any line lying in π has for image a surface generated by a conic in the plane through the line and a variable point of C_4. Any S_3 through π is transformed into itself. Apart from parasitic lines, the only fundamental elements are π and C_4. The former is a proper two

dimensional basis element, whereas C_4 is of dimensionality 1, hence does not appear as part of the base of two V_{10} of the system $|V_{10}|$.

Any S_2 meets π in one point or lies in an S_3 containing π. The order of the surface of parasitic planes (pencils of lines) is found from the intersection of V_{10} with H_6 to be 22. The order of the surface conjugate to an arbitrary plane is then found from $[V_{10}, V_{10}]$ to be 14. From $[V, H(y,\lambda)]$ the 4 remaining units $90 = 64 + 22 + 4$ are accounted for by the planes conjugate to the four points in which S_3 meets C_4. The space $y_5 = 0$ which contains C_4 goes into itself, as its conjugate contains the variety conjugate to C_4. $[H_9(y,\lambda), \Pi_9] = 81 = 56 + 22 + 3$, the 3 being planes conjugate to the three points of C_4 on π. Finally, $[\Pi_9, H_6]$ from $\Pi_9 = H(y,\lambda) - H_6\phi(\lambda)$ has the same value as H_6 with $H(y,\lambda)$. This completes the table. We may now write

$$
\begin{aligned}
&S_3 \sim V_{10}{}^3 : \pi^8 C_4{}^{(3)} \cdot F_{22}\\
&S_2 \sim M_{14}\\
&S_1 \sim C_{10} : [C_{10}, \pi] = 9,\ [C_{10}, C_4] = 9,\ [C_{10}, S_1] = 6\\
&C_4 \sim H_9(y,\lambda) :\ \pi^8 C_9{}^{(2)} F_{22}\\
&\pi \sim \Pi_9 : \pi^7 C_4{}^{(3)} F_{22}\\
&H_6 \sim H_6 : \pi^4 C^{(2)} F_{22}\\
&J_{45} \equiv H^4(y,\lambda) \cdot \Pi_9.
\end{aligned}
$$

This case can be generalized to include the following one. Define π as before, and let C_n be a rational curve of order n, lying on V_3 and having $n-1$ points on π. With obvious changes in ϕ, f_i the equations of the transformation have the same form as before. The general table of characteristics has the form

$$
\begin{aligned}
&S_3 \sim V_{2n+2} :\ \pi^{2n} C_n{}^{(3)} F_{5n+2},\\
&S_2 \sim M_{3n+2},\\
&S_1 \sim C_{2n+2},\ [S_1, C_{2n+2}] = n + 2 :\ [\pi, C_{2n+2}] = 2n + 1 = [C_n, C_{2n+2}],\\
&\pi \sim H(y,\lambda) - H(y)\phi(\lambda) \equiv \Pi_{2n+1} :\ \pi^{2n-1} C_n{}^{(3)} F_{5n+2},\\
&C_n \sim H_{2n+1}(y,\lambda) :\ \pi^{2n} C_n{}^{(2)} F_{5n+2},\\
&H_{n+2} \sim H_{n+2} :\ \pi^n C_n{}^{(2)} F_{5n+2},\\
&J_{10n+5} \equiv H^4(y,\lambda)\Pi_{2n+1}.
\end{aligned}
$$

10. C. Let $l: x_1 = 0,\ x_2 = 0,\ x_3 = 0$ be a line on a general V_3 and R_n be a rational surface of order n, not lying on V_3 but having $n-1$ points on l. A point (y) of S_4 determines a plane (y, l) which meets R_n in one point P. This plane meets V_3 in a residual conic C_2. Let p be the polar line of P as to C_2. The conjugate (y') of the given point (y) in the

harmonic homology P, p generates a birational involutorial transformation of S_4, under which each conic of V_3 in a plane through l is transformed into itself; hence V_3 remains invariant.

The parametric representation of R_n has the form

$$x_i = f_i(r_1, r_2, r_3), \ (i = 1, 2, 3), \qquad x_4 = g_4(r), \qquad x_5 = g_5(r),$$

in which the net $|f(r)|$ is Cremonian, of order n' and $g_4(r) = 0$, $g_5(r) = 0$ contain base points of $|f|$ to multiplicity $n'^2 - n$.

The equation of V_3 has the form

$$V_3\colon\ ux_1 + vx_2 + wx_3 = 0.$$

The plane

$$l,(y)\colon\ x_i = y_i k \qquad\qquad (i = 1, 2, 3),$$

cuts from V_3 the conic

$$k^2 t + k(px_4 + sx_5) + q(x_4, x_5) = 0,$$

in which t is cubic, p and s each quadratic and q linear in y_1, y_2, y_3; q is quadratic in x_4, x_5. If k be replaced by x_1/y_1, the equation of the conic in x_1, x_4, x_5 has coefficients cubic in y_1, y_2, y_3.

The coördinates of P have the form

$$P\colon\ y_1\theta(y), \quad y_2\theta(y), \quad y_3\theta(y), \quad \phi_4(y), \quad \phi_5(y)$$

wherein θ, ϕ_i are obtained as follows: the equations $f_i(r) = y_i$ can be solved for $r_i = f_i^{-1}(y)$, and substituted in $g_i(r)$ and $f_i(r) = y_i F(y)$. The forms $g_k(f^{-1})$ and F contain fundamental curves of orders $n'^2 - n$, so that θ is of order $n - 1$ and each ϕ_k of order n in y_1, y_2, y_3. The polar p of $(y_1\theta, \phi_4, \phi_5)$ as to the conic C_2 is of the form

$$a_1 x_1 + a_4 x_4 + a_5 x_5 = 0,$$

in which a_i is of order $n + 2$ in y_1, y_2, y_3. The factor y_1 has been removed from the equation. The equations of the harmonic homology P, p have the form

$$\begin{aligned} I\colon\quad y_1' &= (\Gamma - 2\theta K)y_1, \quad y_2' = (\Gamma - 2\theta K)y_2, \quad y_3' = (\Gamma - 2\theta K)y_3, \\ y_4' &= \Gamma y_4 - 2K\phi_4, \quad y_5' = \Gamma y_5 - 2K\phi_5. \end{aligned}$$

Here K is the result of substituting y_1, y_4, y_5 for x in the equation of the polar line p of P as to C_2. The factor y_1 can be removed again. $K = 0$ is the variety of invariant points in the involutorial transformation of order $2n + 2$. It is of order $n + 2$ and contains l to multiplicity $n + 1$.

The variety $\Gamma = 0$ is a cone of order $2n + 1$ having l for vertex. Its

equation is obtained by replacing x_1, x_4, x_5 by $y_1\theta$, ϕ_4, ϕ_5 in the equation of the polar p, and removing the factor y_1. It is the projection of the curve γ_{3n}, intersection of R_n, V_3 from l.

The conjugate of the line l is the variety $L=0$ defined by $L \equiv \Gamma - 2\theta K = 0$. It is of order $2n+1$, contains l to multiplicity $2n$, and contains the curve γ_{3n}. Every S_2 through l is transformed into itself.

The base surface M of the system $|V_{2n+2}|$ of varieties conjugate to the $|S_3|$ of S_4 is the complete intersection $\Gamma=0$, $K=0$. It is of order $(2n+1)(n+2)$.

The table of characteristics of I_{2n+2} may be written in the form

$$\begin{aligned}
&S_3 \sim V_{2n+2}: M_{(2n+1).(n+2)}\gamma_{3n}{}^{(2)} \cdot l^{2n+1},\\
&S_2 \sim M_{2n^2+3n+2},\\
&S_1 \sim C_{2n+2}, [C_{2n+2}, l] = 2n+1: [C_{2n+2}, S_1] = 2n+1,\\
&l \sim L_{2n+1}: M_{(2n+1).(n+2)}\gamma_{3n} \cdot l^{2n},\\
&\gamma_{3n} \sim \Gamma_{2n+1}: M_{(2n+1)(n+2)}l^{2n+1}\gamma_{3n},\\
&K_{n+2} \sim K_{n+2}: M_{(2n+1)(n+2)}l^{n+1}\gamma_{3n}.
\end{aligned}$$

Any two V_{2n+2}, varieties conjugate to two S_3 touch each other at every point of γ_{3n}. The conjugate C_{2n+2} of any S_1 lies in the S_3 determined by the given S_1 and l. Any S_3 not containing l meets it in a point. The conjugate V_{2n+2} meets L_{2n+1} in the cubic cone connecting this point with γ_{3n}. The S_3 meets R_n in n lines, each containing 3 points of γ_{3n}. The conjugate V_{2n+2} meets Γ in $M_{(2n+1)(n+2)}$ and in the $3n$ planes connecting these points on γ_{3n} with l. Finally, S_3 meets K_{n+2} in a surface F_{n+2} of invariant points, which is also on its conjugate V_{2n+2}.

The jacobian of the system $|V_{2n+2}|$ is

$$J \equiv L^2{}_{2n+1}\Gamma^3{}_{2n+1}.$$

These results can be generalized immediately to apply to a three dimensional variety V_m of S_4, having l to multiplicity $m-2$. The simplest form of R_n is a plane $x_4=0$, $x_5=0$. An extensive category is that of the monoids in S_3:

$$g_{n-1}(x_1, x_2, x_3)x_4 + h_n(x_1, x_2, x_3) = 0; \quad x_5 = 0.$$

11. A set A', B', C', of generating involutions of another infinite discontinuous Cremona group under which V_3 remains invariant, this time point by point, can be obtained by the central perspective Jonquieres involutions in each plane through l, with P as center and the residual section of the plane with V_3 as curve of invariant points.

THE DERIVATION OF TENSORS FROM TENSOR FUNCTIONS.*

By Alfred K. Mitchell.

Introduction. E. Schroedinger † has given a rule for deriving a tensor from an invariant tensor function. In the first part of the present paper a proof is given of the theorem that if Φ is any invariant function of a tensor its derivative with respect to a component of this tensor is itself a component of a tensor. It is also proved that the derivative of a tensor function of a tensor produces a tensor of higher rank. The first of these theorems is then applied to the invariants of a mixed tensor of rank 2, and the invariants of the tensor so derived are considered. In this way it is seen that if $F_s{}^r$ is any polynomial function $F = a_0 + a_1E + a_2E^2 + \cdots$ of the matrix E, then its invariants can be expressed in terms of the invariants of E and hence the derivatives of the invariants of F are functions of the derivatives of the invariants of E.

1. By a space of n dimensions we mean a continuous arrangement of points; a point being a set of n ordered real numbers $(x^1, x^2, x^3, \cdots, x^n)$.

A set of n equations

$$\bar{x}^r = \bar{x}^r(x^1, x^2, \cdots, x^n) \qquad (r = 1, 2, \cdots, n)$$

in which the functions $\bar{x}^r$ are single valued for all points and which can be solved ‡ so as to yield a set of n equations

$$x^r = x^r(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$$

in which the functions x^r are singled valued, define a transformation of coördinates in the n dimensional space.

A set of $n^{(p+q)}$ functions $X^{r_1r_2\cdots r_p}_{s_1s_2\cdots s_q}$, defined with respect to a coördinate system (x), and from which we obtain, in any other coördinate system $(\bar{x})$, the corresponding $n^{(p+q)}$ functions $\bar{X}^{r_1r_2\cdots r_p}_{s_1s_2\cdots s_q}$, by means of the $n^{(p+q)}$ equations of transformation §

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† See *Annalen der Physik*, Vol. 82 (1927), p. 265.

‡ This implies that the Jacobian determinant $\partial(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)/\partial(x^1, x^2 \cdots x^n)$ is not identically zero. See Goursat-Hedrick, *Mathematical Analysis*, Vol. I, Chap. 2.

§ In these equations and throughout this paper Greek letters occurring as indices in pairs will be used as summation or umbral labels, the summation from 1 to n. See O. Veblen, *Cambridge Tracts*, No. 24, Chaps. 1 and 2.

$$\bar{X}^{r_1r_2\cdots r_p}_{s_1s_2\cdots s_q}=X^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q}\frac{\partial\bar{x}^{r_1}}{\partial x^{\alpha_1}}\frac{\partial\bar{x}^{r_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial\bar{x}^{r_p}}{\partial x^{\alpha_q}}\frac{\partial x^{\beta_1}}{\partial\bar{x}^{s_1}}\frac{\partial x^{\beta_2}}{\partial\bar{x}^{s_2}}\cdots\frac{\partial x^{\beta_q}}{\partial\bar{x}^{s_q}}$$

is said to define a mixed tensor of rank $p+q$ which is contravariant of rank p and covariant of rank q.

Any function X of the coördinates whose value is unchanged when we change the coördinate system is called an invariant or scalar function. The equation of transformation is

$$\bar{X}=X.$$

According to the rule of composition of tensors* we can form an invariant from a contravariant tensor of rank p, $X^{r_1r_2\cdots r_p}$ and a covariant tensor of rank p, $X_{r_1r_2\cdots r_p}$ by forming their scalar product

$$X^{\alpha_1\alpha_2\cdots\alpha_p}X_{\alpha_1\alpha_2\cdots\alpha_p}\ (\alpha_1,\alpha_2\cdots\alpha_p\text{ summation labels}).$$

The converse of the rule of composition of tensors may be stated for the general case as follows. If the functions $X^{a_1\cdots a_p r_1\cdots r_l}_{b_1\cdots b_q s_1\cdots s_m}$ have such a law of transformation that the summation

$$X^{\alpha_1\cdots\alpha_p r_1\cdots r_l}_{\beta_1\cdots\beta_q s_1\cdots s_m}\cdot Y^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}$$

is a mixed tensor, covariant of rank m and contravariant of rank l, where $Y^{t_1\cdots t_q}_{u_1\cdots u_p}$ is an arbitrary mixed tensor contravariant of rank q and covariant of rank p, then the $n^{(p+l+q+m)}$ functions $X^{a_1\cdots a_p r_1\cdots r_l}_{b_1\cdots b_q s_1\cdots s_m}$ form a mixed tensor, contravariant of rank $(p+l)$ and covariant of rank $(q+m)$.

THEOREM 1.† *If* Φ *is any invariant function of a tensor* $E^{r_1\cdots r_p}_{s_1\cdots s_q}$, *contravariant of rank* p, *covariant of rank* q *and of any other tensors which are independent of* $E^{r_1\cdots r_p}_{s_1\cdots s_q}$, *then* $\partial\Phi/\partial E^{r_1\cdots r_p}_{s_1\cdots s_q}$ *is a mixed tensor which is contravariant of rank* q *and covariant of rank* p.

Proof: Let

$$\partial\Phi/\partial E^{r_1\cdots r_p}_{s_1\cdots s_q}=X^{s_1\cdots s_q}_{r_1\cdots r_p}.$$

Then

$$\bar{X}^{s_1\cdots s_q}_{r_1\cdots r_p}=\partial\Phi/\partial\bar{E}^{r_1\cdots r_p}_{s_1\cdots s_q}=\partial\Phi/\partial E^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}\times\partial E^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}/\partial\bar{E}^{r_1\cdots r_p}_{s_1\cdots s_q}.$$

But, by hypothesis,

$$E^{\alpha_1\alpha_2\cdots\alpha_p}_{\beta_1\beta_2\cdots\beta_q}=\bar{E}^{\rho_1\cdots\rho_p}_{\sigma_1\cdots\sigma_q}\frac{\partial x^{\alpha_1}}{\partial\bar{x}^{\rho_1}}\cdots\frac{\partial x^{\alpha_p}}{\partial\bar{x}^{\rho_p}}\frac{\partial\bar{x}^{\sigma_1}}{\partial x^{\beta_1}}\cdots\frac{\partial\bar{x}^{\sigma_q}}{\partial x^{\beta_q}}.$$

* See F. D. Murnaghan, *Vector Analysis and Theory of Relativity*, p. 25.

† The author finds that this theorem is not essentially different from the "Aronhold Process." See Weitzenböck, *Invarianten Theorie*, p. 18.

Hence

$$\partial E^{a_1 \ldots a_p}_{\beta_1 \ldots \beta_q} / \partial \bar{E}^{r_1 \ldots r_p}_{s_1 \ldots s_q} = \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \cdots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Therefore

$$\bar{X}^{s_1 \ldots s_q}_{r_1 \ldots r_p} = X^{\beta_1 \ldots \beta_q}_{a_1 \ldots a_p} \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \cdots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Q. E. D.

Now suppose we have a mixed tensor $X^{r_1 \ldots r_p}_{s_1 \ldots s_q}$ which is contravariant of rank p and covariant of rank q and is a function of a mixed tensor $E^{a_1 \ldots a_l}_{b_1 \ldots b_m}$ and possibly other tensors. Let us form an invariant Φ from the tensor $X^{r_1 \ldots r_p}$ and the arbitrary tensor $Y^{m_1 \ldots m_q}_{n_1 \ldots n_p}$, which is covariant of rank p and contravariant of rank q and which is independent of the tensor $E^{a_1 \ldots a_l}_{b_1 \ldots b_m}$. By the rule of composition, we have

$$\Phi = X^{a_1 \ldots a_p}_{\beta_1 \ldots \beta_q} \cdot Y^{\beta_1 \ldots \beta_q}_{a_1 \ldots a_p},$$

Now by Theorem 1

$$\partial \Phi / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m} = Z^{b_1 \ldots b_m}_{a_1 \ldots a_l},$$

is a mixed tensor contravariant of rank m and covariant of rank l. But

$$\partial \Phi / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m} = (\partial X^{a_1 \ldots a_p}_{\beta_1 \ldots \beta_q} / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m}) Y^{\beta_1 \ldots \beta_q}_{a_1 \ldots a_p} = Z^{b_1 \ldots b_m}_{a_1 \ldots a_l}.$$

Hence, denoting

$$\partial X^{a_1 \ldots a_p}_{\beta_1 \ldots \beta_q} / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m} \quad \text{by} \quad F^{a_1 \ldots a_p b_1 \ldots b_m}_{\beta_1 \ldots \beta_q a_1 \ldots a_l},$$

we have

$$F^{a_1 \ldots a_p b_1 \ldots b_m}_{\beta_1 \ldots \beta_q a_1 \ldots a_l} Y^{\beta_1 \ldots \beta_q}_{a_1 \ldots a_p} = Z^{b_1 \ldots b_m}_{a_1 \ldots a_l},$$

and, therefore, by the converse of the rule of composition of tensors,

$$F^{r_1 \ldots r_p b_1 \ldots b_m}_{s_1 \ldots s_q a_1 \ldots a_l} = \partial X^{r_1 \ldots r_p}_{s_1 \ldots s_q} / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m}$$

is a mixed tensor which is contravariant of rank $(p+m)$ and covariant of rank $(q+l)$. We have therefore proved

THEOREM 2. *If $X^{r_1 \ldots r_p}_{s_1 \ldots s_q}$ is a mixed tensor, contravariant of rank p and covariant of rank q which is a function of a mixed tensor $E^{a_1 \ldots a_l}_{b_1 \ldots b_m}$, and possibly other tensors, then $\partial X^{r_1 \ldots r_p}_{s_1 \ldots s_q} / \partial E^{a_1 \ldots a_l}_{b_1 \ldots b_m}$ is a mixed tensor which is covariant of rank $(q+l)$ and contravariant of rank $(p+m)$.*

2. We shall now apply the first of the above theorems to derive tensors from the invariants of the mixed tensor of rank 2, E_s^r, and shall consider the problem of expressing the invariants of these derived tensors in terms

of the invariants of $E_s{}^r$. Note: The invariants of a covariant tensor of rank 2, E_{rs}, are the invariants of the n-ary quadratic form $f = E_{\alpha\beta}x^\alpha x^\beta$ and there is only one invariant of this form, namely its discriminant.* But $E_s{}^r = g^{r\alpha}E_{\alpha s}$ where $g^{rs} =$ cofactor of $g_{sr} \div |\, g\,|$ and g_{rs} are the coefficients of the differential form $ds^2 = g_{\alpha\beta}\cdot dx^\alpha dx^\beta$, thus the invariants of $E_s{}^r$ may be considered as the simultaneous invariants of the two n-ary quadratic forms $f = E_{\alpha\beta}x^\alpha x^\beta$ and $g = g_{\alpha\beta}x^\alpha x^\beta$. Or, if one prefers, the invariants of $E_s{}^r$ may be regarded as the simultaneous invariants of $E_s{}^r$ and the unit tensor $\delta_s{}^r$ whose components are 1 or zero according as r is equal to s or not.

Introducing the generalized Kronecker delta † $\delta^{r_1\cdots r_k}_{s_1\cdots s_k}$ (which, if the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts, has the value $+1$ or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; and which in all other cases has the value zero) we can form the following invariants of the tensor $E_s{}^r$:

$$I_1 = \delta_\beta{}^\alpha E_\alpha{}^\beta; \qquad I_2 = (1/2\,!)\,\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2},$$

$$I_3 = (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3};$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$I_n = (1/n\,!)\,\delta^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}\cdots E_{\alpha_n}{}^{\beta_n}.$$

Applying Theorem 1 to these invariants, we derive from each of them a mixed tensor. Denote the tensors derived from $I_1, I_2, \cdots I_n$ by $T_{1r}{}^s$, $T_{2r}{}^s, \cdots T_{nr}{}^s$ respectively. Then

$$\begin{aligned}
T_{1r}{}^s &= \partial I_1/\partial E_s{}^r = \delta_r{}^s,\\
T_{2r}{}^s &= \partial I_2/\partial E_s{}^r = \delta^{s\alpha}_{r\beta}E_\alpha{}^\beta = \{\delta_r{}^s\delta_\beta{}^\alpha - \delta_\beta{}^s\delta_r{}^\alpha\}E_\alpha{}^\beta = \delta_r{}^sI_1 - E_r{}^s,\\
T_{3r}{}^s &= \partial I_3/\partial E_s{}^r = (1/2\,!)\,\delta^{s\alpha_1\alpha_2}_{r\beta_1\beta_2}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}\\
&= (1/2\,!)\,\{\delta_r{}^s\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} - \delta_{\beta_1}{}^s\delta^{\alpha_1\alpha_2}_{r\beta_2} - \delta_{\beta_2}{}^s\delta^{\alpha_1\alpha_2}_{\beta_1 r}\}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}\\
&= \delta_r{}^sI_2 - (1/2\,!)\,\{T_{2r}{}^{\alpha_1}E_{\alpha_1}{}^s + T_{2r}{}^{\alpha_2}E_{\alpha_2}{}^s\} = \delta_r{}^sI_2 - T_{2r}{}^\alpha E_\alpha{}^s\\
&= \delta_r{}^sI_2 - (\delta_r{}^\alpha I_1 - E_r{}^\alpha)E_\alpha{}^s = \delta_r{}^sI_2 - I_1E_r{}^s + E_r{}^\alpha E_\alpha{}^s,\\
T_{4r}{}^s &= (1/3\,!)\,\delta^{s\alpha_1\alpha_2\alpha_3}_{r\beta_1\beta_2\beta_3}E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3}\\
&= (1/3\,!)\,\{\delta_r{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} - (\delta_{\beta_1}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{r\beta_2\beta_3} + \delta_{\beta_2}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1 r\beta_3} + \delta_{\beta_3}{}^s\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2 r})\}\,E_{\alpha_1}{}^{\beta_1}E_{\alpha_2}{}^{\beta_2}E_{\alpha_3}{}^{\beta_3}\\
&= \delta_r{}^sI_3 - (1/3)\,\{T_{2r}{}^{\alpha_1}E_{\alpha_1}{}^s + T_{2r}{}^{\alpha_2}E_{\alpha_2}{}^s + T_{2r}{}^{\alpha_3}E_{\alpha_3}{}^s\} = \delta_r{}^sI_3 - T_{2r}{}^\alpha E_\alpha{}^s\\
&= \delta_r{}^sI_3 - I_2E_r{}^s + I_1E_r{}^\alpha E_\alpha{}^s - E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}E_{\alpha_2}{}^s,
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}
T_{pr}{}^s &= \delta_r{}^sI_{p-1} - I_{p-2}E_r{}^s + I_{p-3}E_r{}^\alpha E_\alpha{}^s - I_{p-4}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}E_{\alpha_2}{}^s\\
&\quad + \cdots (-1)^{t+1}I_{p-t}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}\cdots E^s_{\alpha_{t-2}} + \cdots (-1)^{p+1}E_r{}^{\alpha_1}E_{\alpha_1}{}^{\alpha_2}\cdots E^s_{\alpha_{p-2}}.
\end{aligned}$$

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* See Dickson, *Algebraic Invariants*, p. 48.

† See F. D. Murnaghan, *American Mathematical Monthly*, Vol. 32, p. 233.

From each of these mixed tensors we can form invariants analogous to the invariants $I_1, I_2, \cdots I_n$ of the tensor $E_s{}^r$. Denote the invariants formed from

$$\begin{array}{lll} T_{1r}{}^s & \text{by} & I_{11}, I_{21}, \cdots I_{n1}, \\ T_{2r}{}^s & \text{by} & I_{12}, I_{22}, \cdots I_{n2}, \\ T_{3r}{}^s & \text{by} & I_{13}, I_{23}, \cdots I_{n3}, \\ T_{ir}{}^s & \text{by} & I_{1i}, I_{2i}, \cdots I_{ni}. \end{array}$$

For the invariants of $T_{1r}{}^s$, we have

$$\begin{aligned}
I_{11} &= \delta_\beta{}^\alpha \delta_\alpha{}^\beta = \delta_\alpha{}^\alpha = n, \\
I_{21} &= (1/2\,!)\,\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}\,\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2} = (1/2\,!)\,\delta_{\alpha_1\alpha_2} = [n(n-1)/2\,!], \\
I_{31} &= (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\,\delta_{\alpha_1}{}^{\beta_1}\delta_{\alpha_2}{}^{\beta_2}\delta_{\alpha_3}{}^{\beta_3} = (1/3\,!)\,\delta^{\alpha_1\alpha_2\alpha_3}_{\alpha_1\alpha_2\alpha_3} = [n(n-1)(n-2)/3\,!], \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
I_{m1} &= (1/m\,!)\,\delta^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_m}\,\delta_{\alpha_1}{}^{\beta_1}\cdots\delta_{\alpha_m}{}^{\beta_m} = (1/m\,!)\,\delta^{\alpha_1\ldots\alpha_m}_{\alpha_1\ldots\alpha_m} = [n\,!/m\,!(n-m)\,!], \\
I_{n1} &= (n\,!/n\,!) = 1.
\end{aligned}$$

Now denote by $f(\lambda)$ the polynomial

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} + \cdots (-1)^n I_n = f(\lambda),$$

where $I_1, I_2, \cdots I_n$ are the invariants of the tensor $E_r{}^s$. It is evident, from the definition of these invariants (see page 198) and the definition of the generalized Kronecker delta, that I_1 is the sum of the diagonal elements of the matrix $\| E_r{}^s \|$; I_2 is the sum of the two rowed principle minors; I_3 the sum of the three rowed principle minors, etc. and I_n is the determinant of the matrix $\| E_r{}^s \|$. Hence we see that $f(\lambda) = 0$ is the characteristic equation of the matrix $\| E_r{}^s \|$ i. e. the polynomial $f(\lambda)$ is the determinant of the matrix $\| \delta_r{}^s\lambda - E_r{}^s \|$.

Thus the invariants of a mixed tensor $E_r{}^s$ are the coefficients of the characteristic equation of the matrix $\| E_r{}^s \|$.

With this fact and with the theorem from algebra which says that if you have a matrix $\| E_r{}^s \|$, with characteristic equation $f(\lambda) = 0$ and latent roots $\lambda_1, \lambda_2, \cdots \lambda_n$, the latent roots of any polynomial function of $\| E_r{}^s \|$ are the corresponding polynomial functions of the latent roots $\lambda_1, \lambda_2, \cdots \lambda_n$, we shall be able to investigate the nature of the invariants of the tensors $T_{2r}{}^s, T_{3r}{}^s, \cdots T_{pr}{}^s$.

Consider the tensor $T_{2r}{}^s$. Its characteristic polynomial will be the determinant of the matrix $\| \delta_r{}^s\lambda - T_{2r}{}^s \|$. But $T_{2r}{}^s = \delta_r{}^s I_1 - E_r{}^s$, so that the determinant

$$\begin{aligned} | \delta_r{}^s\lambda - T_{2r}{}^s | &= | \delta_r{}^s(\lambda - I_1) + E_r{}^s | \\ &= (-1)^n \, | \delta_r{}^s(I_1 - \lambda) - E_r{}^s | = (-1)^n f(I_1 - \lambda), \end{aligned}$$

where

$$f(\lambda)=\lambda^n-I_1\lambda^{n-1}+I_2\lambda^{n-2}+\cdots(-1)^nI_n=|\,\delta_r{}^s\lambda-E_r{}^s\,|.$$

By Taylor's expansion for a polynomial of degree n,

$$f(I_1-\lambda)=f(I_1)-\lambda f'(I_1)+(\lambda^2/2!)f''(I_1)+\cdots(-1)^n\lambda^n;$$

so that the characteristic polynomial for $T_{2r}{}^s$ is

$$\begin{aligned}\lambda^n&-[f^{(n-1)}(I_1)/(n-1)!]\lambda^{n-1}+[f^{(n-2)}(I_1)/(n-2)!]\lambda^{n-2}\\&+\cdots(-1)^r[f^{(n-r)}(I_1)/(n-r)!]\lambda^{n-r}+\cdots(-1)^nf(I_1),\end{aligned}$$

from which we obtain the invariants of $T_{2r}{}^s$. They are

$$\begin{aligned}
I_{12}&=[f^{(n-1)}(I_1)/(n-1)!]=1/(n-1)!\,\{n!\,I_1-(n-1)!\,I_1\}=(n-1)I_1,\\
I_{22}&=[f^{(n-2)}(I_1)/(n-2)!]\\
&=1/(n-2)!\,\{(n!/2!)I_1{}^2-(n-1)!\,I_1I_1+(n-2)!\,I_2\}\\
&=[n(n-1)/2!]I_1{}^2-(n-1)I_1I_1+I_2,\\
I_{32}&=[f^{(n-3)}(I_1)/(n-3)!]\\
&=1/(n-3)!\,\{(n!/3!)I_1{}^3-[(n-1)!/2!]I_1I_1{}^2\\
&\quad+[(n-2)!/1]I_2I_1-(n-3)!\,I_3\}=[n(n-1)(n-2)/3!]I_1{}^3\\
&\quad-[(n-1)(n-2)/2!]I_1I_1{}^2-(n-2)I_2I_1-I_3,\\
I_{s2}&=[f^{(n-s)}(I_1)/(n-s)!]=1/(n-s)!\,\{(n!/s!)I_1{}^s\\
&\quad-[(n-1)!/(s-1)!]I_1I_1{}^{s-1}+[(n-2)!/(s-2)!]I_2I_1{}^{s-2}\\
&\quad+\cdots(-1)^r[(n-r)!/(s-r)!]I_rI_1{}^{s-r}+\cdots(-1)^s(n-s)!\,I_s\}\\
&=[n!/(n-s)!\,s!]I_1{}^s-[(n-1)!/(n-s)!\,(s-1)!]I_1I_1{}^{s-1}\\
&\quad+[(n-2)!/(n-s)!\,(s-2)!]I_2I_1{}^{s-2}\\
&\quad+\cdots(-1)^r[(n-r)!/(n-s)!\,(s-r)!]I_rI_1{}^{s-r}+\cdots(-1)^sI_s,\\
I_{n2}&=f(I_1)=I_1{}^n-I_1I_1{}^{n-1}+I_2I_1{}^{n-2}+\cdots(-1)^rI_rI_1{}^{n-r}+\cdots(-1)^nI_n.
\end{aligned}$$

Likewise the invariants $I_{13}, I_{23},\cdots I_{n3}$ of the tensor $T_{3r}{}^s$ will be the coefficients of the characteristic equation of the matrix $\|\,T_{3r}{}^s\,\|$ and since (from the expression for $T_{3r}{}^s$ on page 198)

$$\|\,T_{3r}{}^s\,\|=\|\,\delta_r{}^sI_2\,\|-I_1\|\,E_r{}^s\,\|+\|\,E_r{}^s\,\|^2$$

by the theorem stated above, $I_{13}, I_{23},\cdots I_{n3}$ will be the coefficients of the equation whose roots are $(I_2-I_1\lambda_1+\lambda_1{}^2)$, $(I_2-I_1\lambda_2+\lambda_2{}^2),\cdots$ $(I_2-I_1\lambda_n+\lambda_n{}^2)$ where $\lambda_1, \lambda_2\cdots\lambda_n$ are the roots of

$$f(\lambda)=\lambda^n-I_1\lambda^{n-1}+I_2\lambda^{n-2}+\cdots(-1)^nI_n=0.$$

From the elementary theory of equations we see that for this last equation

$$I_1 = \Sigma\lambda_1$$
$$I_2 = \Sigma\lambda_1\lambda_2$$
$$\cdots\cdots\cdots$$
$$I_n = \Sigma\lambda_1\lambda_2\cdots\lambda_n.$$

Similarly, since $I_{13}, I_{23}, \cdots I_{n3}$ play a corresponding rôle in the characteristic equation of the matrix $\| T_{3r}{}^s \|$, we shall have

$$I_{13} = \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2),$$
$$I_{23} = \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2),$$
$$I_{33} = \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2)(I_2 - I_1\lambda_3 + \lambda_3^2), \quad \text{etc.}$$

By multiplying out the factors of a term in the above summations we find

$$\begin{aligned}
I_{13} &= nI_2 - I_1\Sigma\lambda_1 + \Sigma\lambda_1^2,\\
I_{23} &= [n(n-1)/2!]I_2^2 - (n-1)I_2I_1\Sigma\lambda_1\\
&\quad + (n-1)I_2\Sigma\lambda_1^2 + I_1^2\Sigma\lambda_1\lambda_2 - I_1\Sigma\lambda_1\lambda_2^2 + \Sigma\lambda_1^2\lambda_2^2\\
&= [n(n-1)/2!]I_2^2 - (n-1)I_2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} + \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\},\\
I_{33} &= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2I_1\Sigma\lambda_1\\
&\quad + [(n-1)(n-2)/2!]I_2^2\Sigma\lambda_1^2 + (n-2)I_2I_1^2\Sigma\lambda_1\lambda_2\\
&\quad - (n-2)I_2I_1\Sigma\lambda_1\lambda_2^2 + (n-2)I_2\Sigma\lambda_1^2\lambda_2^2\\
&\quad + \Sigma\lambda_1^2\lambda_2^2\lambda_3^2 - I_1\Sigma\lambda_1\lambda_2^2\lambda_3^2 + I_1^2\Sigma\lambda_1\lambda_2\lambda_3^2 - I_1^3\Sigma\lambda_1\lambda_2\lambda_3\\
&= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\}\\
&\quad + (n-2)I_2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\}\\
&\quad - \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)(I_1\lambda_3 - \lambda_3^2)\}.
\end{aligned}$$

From which we see that

$$\begin{aligned}
I_{43} &= [n(n-1)(n-2)(n-3)/4!]I_2^4\\
&\quad - [(n-1)(n-2)(n-3)/3!]I_2^3\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\}\\
&\quad + [(n-2)(n-3)/2!]I_2^2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\}\\
&\quad - (n-3)I_2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)(I_1\lambda_3 - \lambda_3^2)\}\\
&\quad + \{\Sigma(I_1\lambda_1 - \lambda_1^2)(\quad)\cdots(I_1\lambda_4 - \lambda_4^2)\},
\end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}
I_{r3} &= [n!/(n-r)!\,r!]I_2^r - [(n-1)!/(n-r)!\,(r-1)!]I_2^{r-1}\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\}\\
&\quad + \cdots (-1)^{s-1}[(n-s+1)!/(n-r)!\,(r-s+1)!]I_2^{r-s+1}\\
&\quad \times \{\Sigma(I_1\lambda_1 - \lambda_1^2)\cdots(I_1\lambda_{s-1} - \lambda^2_{s-1})\}\\
&\quad + \cdots (-1)^r\{\Sigma(I_1\lambda_1 - \lambda_1^2)\cdots(I_1\lambda_r - \lambda_r^2)\}.
\end{aligned}$$

Substituting $I_1 = \Sigma\lambda_1$ and $\Sigma\lambda_1^2 = I_1^2 - 2I_2$,

$$I_{13} = nI_2 - I_1^2 + I_1^2 - 2I_2 = (n-2)I_2.$$

Furthermore it is easily seen that

$$\Sigma\lambda_1\lambda_2^2 = I_1I_2 - 3I_3, \quad \Sigma\lambda_1^2\lambda_2^2 = I_2^2 - 2I_1I_3 + 2I_4, \quad \Sigma\lambda_1\lambda_2 = I_2.$$

Substituting these results in the expression (page 201) for I_{23} we obtain

$$I_{23} = [n(n-1)/2!]I_2^2 - 2(n-1)I_2^2 + I_2^2 + I_1 I_3 + 2I_4.$$

Since $\Sigma\lambda_1^2\lambda_2^2\lambda_3^2$ is of degree 6 in the λ's, it is of weight six in the coefficients I, and we write

$$\Sigma\lambda_1^2\lambda_2^2\lambda_3^2 = AI_2I_5 + BI_2I_4 + CI_3^2 + DI_6.$$

Let

$$\lambda_1 = \lambda_2 = \lambda_3 = 1; \qquad \lambda_4 = \lambda_5 = \cdots = \lambda_n = 0.$$

Then

$$I_1 = 3, \quad I_2 = 3, \quad I = 1, \quad I_4 = I_5 = 0.$$

Substituting we obtain $C = 1$.

Next let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$; $\lambda_5 = \cdots = \lambda_n = 0$. Then

$$I_1 = 4, \quad I_2 = 6, \quad I_3 = 4, \quad I_4 = 1, \quad I_5 = I_6 = 0.$$

We get $B = -2$.

Continuing in this way we find that

$$\Sigma\lambda_1^2\lambda_2^2\lambda_3^2 = 2I_1I_5 - 2I_2I_4 + I_3^2 - 2I_6.$$

And by a similar process we obtain

$$\Sigma\lambda_1\lambda_2^2\lambda_3^2 = I_2I_3 - 3I_1I_4 + 5I_5, \quad \Sigma\lambda_1\lambda_2\lambda_3^2 = I_1I_3 - 4I_4, \quad \Sigma\lambda_1\lambda_2\lambda_3 = I_3.$$

Substituting these and the foregoing results in the expression (page 13) for I_{33}, we find

$$\begin{aligned} I_{33} = {} & [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2(2I_2) \\ & + (n-2)I_2(I_2^2 + I_1I_3 + 2I_4) \\ & - (2I_6 + I_1I_2I_3 + I_1^2I_4 + 3I_1I_5 + 2I_2I_4 + I_3^2). \end{aligned}$$

We observe from the expression for I_{r3} on page 201 that this invariant is a polynomial of degree r in I_2 and that the coefficients of this polynomial are symmetric functions of the roots of $f(\lambda) = 0$. These symmetric functions being of the form $\Sigma\lambda_1^p\lambda_2^q\lambda_3^r \cdots \lambda_s^t$ where the exponents are at most 2, can be calculated by means of known * formulae in terms of the coefficients of $f(\lambda) = 0$, i.e. in terms of the invariants $I_1, I_2, \cdots I_n$. It is evident from the foregoing expressions for the I_{13}, I_{23}, I_{33}, however, that the expression for I_{r3} is not conveniently put explicitly in terms of I_1, I_2, etc.

Finally we observe from the expressions (page 198) for the tensors T_r^s that each of the matrices $\| T_r^s \|$ is a polynomial function of the matrix $\| E_r^s \|$.

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* See Burnside and Panton, *Theory of Equations*, p. 269.

Thus

$$\|T_{2r}{}^{s}\| = \|\delta_r{}^s I_1\| - \|E_r{}^s\|,$$
$$\|T_{3r}{}^{s}\| = \|\delta_r{}^s I_2\| - I_1\|E_r{}^s\| + \|E_r{}^s\|^2,$$
$$\|T_{4r}{}^{s}\| = \|\delta_r{}^s I_3\| - I_2\|E_r{}^s\| + I_1\|E_r{}^s\|^2 - \|E_r{}^s\|^3, \quad \text{etc.}$$

It follows from the theorem stated on page 199 that the latent roots of each of these matrices are the corresponding polynomial functions of the roots of $f(\lambda) = 0$. Now given an equation of degree n, $f(\lambda) = 0$ whose roots are $\lambda_1, \lambda_2 \cdots \lambda_n$ we can, at least theoretically, by means of Tschirnhausen's transformation,* form the equation whose roots are any polynomial function of degree $(n-1)$ or less of the roots of $f(\lambda) = 0$, and the coefficients of the transformed equation will be given in terms of the coefficients of $f(\lambda) = 0$. Thus the invariants (coefficients of the characteristic equation of the matrix $\|T_{pr}{}^s\|$) of any one of the tensors $T_{pr}{}^s$ are functions of the invariants (coefficients of the equation $f(\lambda) = 0$) of the tensor $E_r{}^s$.

In general we can say, as is evident from the foregoing considerations, if $F_s{}^r$ is any polynomial function

$$F = a_0 + a_1 E + a_2 E^2 + \cdots,$$

of the matrix E then its invariants can be expressed in terms of the invariants of E, and hence the derivatives, with respect to $E_s{}^r$, of its invariants can be expressed in terms of the tensors $T_{1r}{}^s, T_{2r}{}^s \cdots T_{nr}{}^s$.

* See Burnside and Panton, *op. cit.*, p. 318.

THE BEHAVIOR OF MEAN-SQUARE OSCILLATION AND CONVERGENCE UNDER REGULAR TRANSFORMATIONS.*

By Ralph Palmer Agnew.

1. *Introduction.* Recently the writer † has considered the behavior of continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of complex and real sequences of functions under complex and real regular transformations with triangular matrices. It is the object of this paper to extend that investigation to consider the behavior of mean square oscillation and convergence in the mean of sequences of complex and of real measurable functions under complex and real regular transformations with triangular matrices.

2. *Transformations.* We recall that a transformation with a triangular matrix is a sequence-to-sequence transformation of the form

$$\sigma_n = a_{n1}s_1 + a_{n2}s_2 + \cdots + a_{nn}s_n, \qquad (n = 1, 2, 3, \cdots),$$

where the a_{nk} are constants, and that a transformation is said to be *regular* when it carries every convergent sequence $\{s_n\}$ into a sequence $\{\sigma_n\}$ which converges to the same value. Such a transformation is said to be *real* when a_{nk} is real for all n and k; otherwise it is complex. The transformations and sequences considered in this paper may, except in cases where a specific statement to the contrary is made, be complex.

The following six conditions which are to be used in this paper, are listed together for convenience:

C_1: $\sum_{k=1}^{n} | a_{nk} |$ is bounded for all n;

C_2: for each k, $\lim_{n\to\infty} a_{nk} = 0$;

C_3: $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$;

C_4: $\lim_{n\to\infty} \sum_{k=1}^{n} | a_{nk} | = 1$;

C_5: for each k, $a_{nk} = 0$ for almost ‡ all n;

C_6: $\sum_{k=1}^{n} a_{nk} = 1$ for almost all n.

* Presented to the American Mathematical Society, April 18, 1930, together with the paper referred to in § 1.

† "The Behavior of Bounds and Oscillations of Sequences of Functions under Regular Transformations," *Transactions of the American Mathematical Society*, Vol. 32 (1930), pp. 669-708.

‡ i. e. $a_{nk} = 0$ except for at most a finite number, depending on k, of values of n.

Let the symbol (T) represent a regular transformation with a triangular matrix. By the Silverman-Toeplitz theorem, C_1, C_2, and C_3 are necessary and sufficient for the regularity of a complex transformation when applied to complex sequences, and of a real transformation when applied to real sequences. Hence (T), complex or real, satisfies C_1, C_2, and C_3.

A property * of regular transformations which we shall have occasion to use is included in the following

LEMMA 2. 1. *If (T) fails to satisfy C_4, then there is a bounded sequence $\{s_n\}$ of constants such that*

$$\limsup_{m\to\infty,\, n\to\infty} |\sigma_m - \sigma_n| > \limsup_{m\to\infty,\, n\to\infty} |s_m - s_n|;$$

if (T) is real, $\{s_n\}$ may be taken real.

3. *Mean Square Oscillation and Convergence in the Mean.* Let a sequence $\{f_n(x)\}$ of measurable functions † be defined over a set A in a Euclidean space, and let the Lebesgue integral

$$\int_A |f_m(x) - f_n(x)|^2\, dx$$

exist for all sufficiently great values of m and n; then

$$\mathfrak{M}(\{f_n\}, A) = \limsup_{m\to\infty,\, n\to\infty} \int_A |f_m(x) - f_n(x)|^2\, dx$$

may be called the *mean square oscillation* of $\{f_n(x)\}$ over A.

For the purpose of relating mean square oscillation to the well known concept of convergence in the mean, we shall give some lemmas.

LEMMA 3. 1. *In order that a measurable function $f(x)$ may be summable,‡ it is necessary and sufficient that $|f(x)|$ be summable.*§

The following lemma is easily established.

LEMMA 3. 2. *In order that a measurable function $f(x) = u(x) + iv(x)$, u and v being real, may be of summable square, it is necessary and sufficient that u and v be of summable square.*

Using this lemma, it can readily be shown that the sums and differences of measurable complex functions of summable square are measurable func-

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* W. A. Hurwitz, *American Journal of Mathematics*, Vol. 52 (1930), pp. 611-616.

† A complex function $f(x) = u(x) + iv(x)$, u and v being real, is said to be measurable when u and v are measurable.

‡ A complex function $f(x) = u(x) + iv(x)$ is said to be summable (to the value $\int u dx + i \int v dx$) when u and v are summable.

§ For real functions, this is a standard result. Caratheodory, *Vorlesungen über Reellen Funktionen* (1927), p. 434.

tions of summable square, and that if two measurable complex functions are of summable square, then their product is summable.

By the Riesz-Fischer theorem * if $f_n(x)$ is real, measurable, and of summable square for all (or almost all) values of n and $\mathfrak{M}(\{f_n\}, A) = 0$, there is a measurable function $f(x)$ of summable square, uniquely determined except over a set of measure 0, such that

$$\lim_{n\to\infty} \int_A | f(x) - f_n(x) |^2 \, dx = 0;$$

then $\{f_n(x)\}$ is said to converge in the mean to $f(x)$. We may show that sequences of complex functions have the same property by proving

Lemma 3. 3. *If $f_n(x)$ is measurable and of summable square for almost all values of n and $\mathfrak{M}(\{f_n\}, A) = 0$, then there is a measurable function $f(x)$ of summable square, uniquely determined except over a set of measure* 0, *such that*

$$\lim_{n\to\infty} \int_A | f(x) - f_n(x) |^2 \, dx = 0.$$

Obviously the condition $f_n(x)$ is measurable and of summable square for almost all n is a sufficient but not a necessary condition for the existence, finite or infinite, of $\mathfrak{M}(\{f_n\}, A)$; it follows at once from the definition of $\mathfrak{M}(\{f_n\}, A)$ and from lemma 3. 1 that, for sequences of measurable functions, a necessary and sufficient condition is that $f_m(x) - f_n(x)$ shall be of summable square for all sufficiently great m and n. We shall show that a sequence $\{f_n(x)\}$ of measurable functions converges in the mean to a measurable function whenever its mean square oscillation is zero by proving

Lemma 3. 4. *If $\{f_n(x)\}$ is a sequence of measurable functions and $\mathfrak{M}(\{f_n\}, A) = 0$, then there is a measurable function $f(x)$, uniquely determined except over a set of measure* 0, *such that $f(x) - f_n(x)$ is of summable square for almost all n and*

$$\lim_{n\to\infty} \int_A | f(x) - f_n(x) |^2 \, dx = 0.$$

Choose an index p such that $f_m(x) - f_n(x)$ is of summable square for $m \geqq p$ and $n \geqq p$, and let $\phi_n(x) = f_n(x) - f_p(x)$. Then $\phi_n(x)$ is measurable and of summable square for $n \geqq p$ and $\phi_m - \phi_n = f_m - f_n$ so that $\mathfrak{M}(\{\phi_n\}, A) = 0$; hence, applying the preceding lemma, there is a measurable function $\phi(x)$ of summable square, uniquely determined except over a set of measure 0, such that

* H. Weyl, *Mathematische Annalen*, Vol. 68 (1910), p. 242.

$$\lim_{n\to\infty} \int_A | \phi(x) - \phi_n(x) |^2 dx = 0,$$

or

$$\lim_{n\to\infty} \int_A | \phi(x) + f_p(x) - f_n(x) |^2 dx = 0.$$

It is easily seen that $f(x) = \phi(x) + f_p(x)$ is the required function. It may be noted that, since $\phi(x)$ is measurable and of summable square and $f_p(x)$ is measurable, $f(x)$ is or is not of summable square according as f_p is or is not of summable square.

4. *Oscillation when $s_n(x)$ is of Summable Square for All n.* The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of which each element is measurable and of summable square.

THEOREM 4. 1. *In order that (T) may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A) \tag{4.11}$$

for every sequence $\{s_n(x)\}$, defined over a set A of measure $m(A) > 0$, such that $s_n(x)$ is measurable and of summable square over A for all n, C_4 is necessary and sufficient.

C_4 is necessary; for when C_4 is denied the bounded sequence of constants of lemma 2. 1 contradicts (4. 11). If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required for (4. 11) is automatically satisfied. If $\mathfrak{M}(\{s_n\}. A)$ is finite, let q be any greater number; then there is an index p such that

$$\int_A | s_\mu(x) - s_\nu(x) |^2 dx < q \quad \text{for} \quad \mu \geqq p,\ \nu \geqq p. \tag{4.12}$$

It follows from (4. 12) and Schwarz's inequality that

$$\left[\left(\int_A | s_n(x) |^2 dx\right)^{1/2} - \left(\int_A | s_p(x) |^2 dx\right)^{1/2}\right]^2 < q \quad \text{for} \quad n \geqq p;$$

hence there is a constant, say Q, such that

$$\int_A | s_n(x) |^2 dx \leqq Q \quad \text{for} \quad n \geqq p. \tag{4.121}$$

Since $s_n(x)$ is of summable square for $n = 1, 2, 3, \cdots, p$ there is a constant, say R, such that

$$\int_A | s_n(x) |^2 dx \leqq R \quad \text{for all } n;$$

thus

$$\int_A | s_m(x) | \, | s_n(x) | dx \leqq R \quad \text{for all } m \text{ and } n. \tag{4.122}$$

Again it follows from (4.12) and Schwarz's inequality that

$$(4.13) \qquad \int_A |s_\mu(x) - s_\nu(x)| \, |s_\xi(x) - s_\eta(x)| \, dx < q$$
$$\text{for} \quad \mu \geqq p, \ \nu \geqq p, \ \xi \geqq p, \ \eta \geqq p.$$

We may write for $m > p$, $n > p$,

$$\sigma_m(x) - \sigma_n(x) = \sum_{k=1}^{m} a_{mk} s_k(x) - \sum_{k=1}^{n} a_{nk} s_k(x)$$
$$= \sum_{k=1}^{p} a_{mk} s_k - \sum_{k=1}^{p} a_{nk} s_k + \sum_{\mu=p+1}^{m} a_{m\mu} s_\mu - \sum_{\nu=p+1}^{n} a_{n\nu} s_\nu,$$

and obtain the identity

$$(4.14) \qquad \sigma_m(x) - \sigma_n(x) = \sum_{k=1}^{p} a_{mk} s_k - \sum_{k=1}^{p} a_{nk} s_k$$
$$+ \left(\sum_{\mu=p+1}^{m} a_{m\mu} s_\mu \right) \left(1 - \sum_{\nu=p+1}^{n} a_{n\nu} \right)$$
$$- \left(\sum_{\nu=p+1}^{n} a_{n\nu} s_\nu \right) \left(1 - \sum_{\mu=p+1}^{m} a_{m\mu} \right) + \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu).$$

Thus

$$\mathfrak{M}(\sigma_n, A) = \limsup_{m\to\infty,\, n\to\infty} \int_A |\sigma_m(x) - \sigma_n(x)|^2 \, dx$$
$$(4.15) \qquad = \limsup_{m\to\infty,\, n\to\infty} \int_A |F_1 + F_2 + F_3 + F_4 + F_5|^2 \, dx$$
$$\leqq \limsup_{m\to\infty,\, n\to\infty} \int_A (|F_1| + |F_2| + |F_3| + |F_4| + |F_5|)^2 \, dx,$$

where F_1, F_2, F_3, F_4, and F_5 are in order the terms of the right member of (4.14). For $m > p$, $n > p$ we may write

$$\int_A |F_5|^2 \, dx \leqq \int_A \left[\sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} |a_{m\mu}| \, |a_{n\nu}| \, |s_\mu(x) - s_\nu(x)| \right]^2 dx$$
$$\leqq \int_A \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} \sum_{\xi=p+1}^{m} \sum_{\eta=p+1}^{n} |a_{m\mu}| \, |a_{n\nu}| \, |a_{m\xi}| \, |a_{n_\eta}| \, |s_\mu - s_\nu| \, |s_\xi - s \ | \, dx;$$

and using (4.13) we obtain for $m > p$, $n > p$

$$\int_A |F_5|^2 \, dx \leqq \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} \sum_{\xi=p+1}^{m} \sum_{\eta=p+1}^{n} |a_{m\mu}| \, |a_{n\nu}| \, |a_m| \, |a_n{}^\eta| \, q \leqq q B_m^2 B_n^2$$

where

$$B_n = \sum_{k=1}^{n} |a_{nk}|$$

so that

$$(4.16) \qquad \limsup_{m\to\infty,\, n\to\infty} \int_A |F_5|^2 \, dx \leqq \limsup_{m\to\infty,\, n\to\infty} (q B_m^2 B_n^2).$$

Employing (4.122) and the regularity of (T), methods similar to that by which (4.16) was obtained suffice to prove that

$$(4.17) \qquad \lim_{m\to\infty,\, n\to\infty} \int_A |F_r|^2 dx = 0, \qquad (r = 1, 2, 3, 4).$$

The right member of (4.16) being finite by C_1, it follows from (4.16), (4.17), and Schwarz's inequality that

$$(4.18) \qquad \lim_{m\to\infty,\, n\to\infty} \int_A |F_r| \, |F_\rho| \, dx = 0,$$

$$(r = 1, 2, \cdots, 5;\ \rho = 1, 2, \cdots, 5, \text{ except when } r = \rho = 5).$$

Using (4.16) and (4.18), we find from (4.15) that

$$(4.19) \qquad \mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Using, for the first time,* C_4 we obtain $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$; and since q is any number greater than $\mathfrak{M}(\{s_n\}, A)$, (4.11) follows and the theorem is proved.

The same proof establishes the three theorems 4.2, 4.3, and 4.4 outlined in the summary of § 8.

THEOREM 4.5. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A, *such that* $s_n(x)$ *is measurable and of summable square over* A *for all* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, *no further conditions need be imposed on* a_{nk}.

To prove this theorem, we may choose an arbitrarily small positive number q and proceed exactly as in the sufficiency proof of theorem 4.1 to obtain 4.19. Thus $\mathfrak{M}(\{\sigma_n\}, A) \leqq qB^4$ where $B = \limsup B_n$; and since B is finite by C_1, qB^4 is arbitrarily small, $\mathfrak{M}(\{\sigma_n\}, A) = 0$ and the theorem is proved.

Using the properties of regular transformations and properties of measurable functions of summable square, we obtain from the preceding theorem the

THEOREM 4.6. *Any regular transformation with a triangular matrix carries any sequence of measurable functions of summable square, which converges in the mean over a given set, into a sequence of measurable functions of summable square which converges in the mean over the set to the same function.*

5. *Oscillation when* $s_n(x)$ *is of Summable Square for Almost All* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of which all sufficiently advanced elements are measurable and of summable square.

*** The use of C_4 has been delayed at a slight inconvenience so that the preceding work may furnish the basis of the proofs of later theorems.**

THEOREM 5.1. *In order that (T) may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A) \tag{5.11}$$

for every sequence $\{s_n(x)\}$, defined over a set A of measure $m(A) > 0$, such that $s_n(x)$ is measurable and of summable square over A for almost all n, C_4 *and* C_5 *are necessary and sufficient.*

The necessity of C_4 follows from lemma 2.1. To show that C_5 is necessary, we shall suppose that (T) fails to satisfy C_5 and construct an admissible real sequence $\{s_n(x)\}$, bounded above (below) over A for all n, such that $\mathfrak{M}(\{s_n\}, A) = 0$ and such that the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails. From a denial of C_5 it follows that there is a value of k, say λ, and a sequence $\{n_\alpha\}$ of indices such that $\lim n_\alpha = +\infty$ and $a_{n_\alpha,\lambda} \neq 0$ for $\alpha = 1, 2, 3, \cdots$; furthermore owing to C_2 the sequence $\{n_\alpha\}$ may be so chosen that $|a_{n_i,\lambda}| \neq |a_{n_j,\lambda}|$ when $i \neq j$. Let $\{A_\alpha\}$ be a sequence of subsets of A such that no two subsets have points in common and such that $m(A_\alpha) = m(A)/2^{2\alpha}$, $\alpha = 1, 2, 3, \cdots$.* Define the sequence $\{s_n(x)\}$ as follows: $s_n(x) = 0$ over A for $n \neq \lambda$; $s_\lambda(x) = (-1)^h 2^\alpha / |a_{n_\alpha,\lambda}|$ over A_α, $\alpha = 1, 2, 3, \cdots$, and $s_\lambda(x) = 0$ for all remaining points x of A. Evidently $s_n(x)$ is bounded above or below over A for all n according as $h = 1$ or 2 and is measurable and of summable square for almost all n, and $\mathfrak{M}(\{s_n\}, A) = 0$. But for $n_i > \lambda$, $n_j > \lambda$, and x in A_α,

$$|\sigma_{n_i}(x)| = 2^\alpha |a_{n_i,\lambda}| / |a_{n_\alpha,\lambda}| \quad \text{and} \quad |\sigma_{n_j}(x)| = 2^\alpha |a_{n_j,\lambda}| / |a_{n_\alpha,\lambda}|$$

so that

$$|\sigma_{n_i}(x) - \sigma_{n_j}(x)| \geqq 2^\alpha \big|\, |a_{n_i,\lambda}| - |a_{n_j,\lambda}| \,\big| / |a_{n_\alpha,\lambda}| \quad \text{over } A_\alpha$$

and

$$\begin{aligned}\int_{A_\alpha} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2\, dx &\geqq 2^{2\alpha}[\, |a_{n_i,\lambda}| - |a_{n_j,\lambda}| \,]^2 m(A_\alpha)/|a_{n_\alpha,\lambda}|^2 \\ &\geqq m(A)[\, |a_{n_i,\lambda}| - |a_{n_j,\lambda}| \,]^2/|a_{n_\alpha,\lambda}|^2.\end{aligned}$$

From the preceding inequality and C_2 we obtain for $n_i > \lambda$, $n_j > \lambda$ and $i \neq j$

$$\lim_{p\to\infty} \sum_{\alpha=1}^{p} \int_{A_\alpha} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2\, dx = +\infty;$$

hence $\int_A |\sigma_m(x) - \sigma_n(x)|^2\, dx$ does not exist for all sufficiently large values of m and n, $\mathfrak{M}(\{\sigma_n\}, A)$ does not exist,† the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails, and necessity of C_5 is established.

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* The possibility of subdividing A in this manner is assured by the fact that the measure of the set of points of A, which lie within a "sphere" with a fixed center and radius r, is a continuous monotonically increasing function of r.

† If one cares to admit $+\infty$ as a value of a Lebesgue integral, and to consider sequences of which elements are $+\infty$, then $\mathfrak{M}(\{\sigma_n\}, A)$ is $+\infty$.

If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number. Choose an index p so great that $s_n(x)$ is measurable and of summable square for $n \geqq p$ and also so great that (4.12) holds and obtain (4.13). We may obtain (4.121) and, using Schwarz's inequality,

$$(5.12) \qquad \int_A |s_m(x)| \; |s_n(x)| \, dx \leqq Q \quad \text{for} \quad m \geqq p, \; n \geqq p.$$

Considering (4.15) we can, owing to C_5, choose an index $N \geqq p$ so great that $F_1 = F_2 = 0$ for $m \geqq N$, $n \geqq N$; hence

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} \int_A (|F_3| + |F_4| + |F_5|)^2 \, dx.$$

Using (4.13), (5.12) and the methods of the proof of theorem 4.1, we find that

$$\limsup_{m\to\infty,\, n\to\infty} \int_A |F_5|^2 \, dx \leqq \limsup_{m\to\infty,\; n\to\infty} (qB_m^2 B_n^2).$$

and

$$\lim_{m\to\infty,\, n\to\infty} \int_A |F_r| \; |F_\rho| \, dx = 0 \text{ for } r = 3, 4, 5; \; \rho = 3, 4, 5, \text{ except when } r = \rho = 5,$$

and hence that

$$(5.13) \qquad \mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Finally, using C_4, we have $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and the theorem follows. The same proof establishes the two theorems 5.2 and 5.3 of the summary.

THEOREM 5.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_s(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable and of summable square over* A *for almost all* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, C_5 *is necessary and sufficient.*

Necessity is established exactly as in theorem 5.1. The sufficiency proof is a modification of that of theorem 5.1 in the same sense that the proof of theorem 4.5 is a modification of the sufficiency proof of theorem 4.1. The same proof establishes theorems 5.5 and 5.6 of the summary.

6. *Oscillation when* $s_m(x) - s_n(x)$ *is of Summable Square for All* m *and* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of sequences of measurable functions such that the difference of any two functions of the sequence is of summable square.

THEOREM 6.1. *In order that* (T) *may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable for all* n *and* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n, C_4 *and* C_6 *are necessary and sufficient.*

The necessity of C_4 follows from lemma 2.1. To show that C_6 is necessary, we shall suppose that (T) fails to satisfy C_6 and construct an admissible real sequence $\{s_n(x)\}$, bounded above (below) over A for all n, such that $\mathfrak{M}(\{s_n\}, A) = 0$ and the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails. From a denial of C_6 it follows that there is a sequence $\{n_\alpha\}$ of indices such that

$$\lim_{\alpha\to\infty} n_\alpha = +\infty \text{ and } \Big| \sum_{k=1}^{n_\alpha} a_{n_\alpha,k} - 1 \Big| \neq 0 \qquad \text{for} \quad \alpha = 1, 2, 3, \cdots;$$

furthermore owing to C_3 the sequence $\{n_\alpha\}$ may be so chosen that

$$\Big| \sum_{k=1}^{n_i} a_{n_ik} - 1 \Big| \neq \Big| \sum_{k=1}^{n_j} a_{n_jk} - 1 \Big| \qquad \text{when} \quad i \neq j.$$

Let $\{A_\alpha\}$ be a sequence of subsets of A such that no two subsets have points in common and such that $m(A_\alpha) = m(A)/2^{2\alpha}$ for $\alpha = 1, 2, 3, \cdots$. Let a function $s(x)$ be defined as follows:

$$s(x) = (-1)^h 2^\alpha / \Big| 1 - \sum_{k=1}^{n_\alpha} a_{n_\alpha k} \Big| \text{ over } A_\alpha, \qquad \alpha = 1, 2, 3, \cdots,$$

and $s(x) = 0$ for all other x in A; and let $s_n(x) = s(x)$ for all n. Evidently $s_n(x)$ is measurable over A for all n and is bounded above or below over A for all n according as $h = 1$ or 2, $s_m(x) - s_n(x)$ is of summable square for all m and n, and $\mathfrak{M}(\{s_n\}, A) = 0$. But for x in A

$$| \sigma_{n_i}(x) - \sigma_{n_j}(x) | \geqq | s(x) | \; \Big| \; \Big| 1 - \sum_{k=1}^{n_i} a_{n_ik} \Big| - \Big| 1 - \sum_{k=1}^{n_j} a_{n_jk} \Big| \; \Big|$$

and hence for x in A_α

$$| \sigma_{n_i}(x) - \sigma_{n_j}(x) | \geqq 2^\alpha \; \Big| \; \Big| 1 - \sum_{k=1}^{n_i} a_{n_ik} \Big| - \Big| 1 - \sum_{k=1}^{n_j} a_{n_jk} \Big| \; \Big| / \Big| 1 - \sum_{k=1}^{n_\alpha} a_{n_\alpha k} \Big|$$

and

$$\int_{A_\alpha} |\sigma_{n_i}(x) - \sigma_{n_j}(x)|^2 \, dx \geqq m(A) \Big[\Big|1 - \sum_{k=1}^{n_i} a_{n_ik}\Big| - \Big|1 - \sum_{k=1}^{n_j} a_{n_jk}\Big| \Big]^2 / \Big|1 - \sum_{k=1}^{n_\alpha} a_{n_\alpha k}\Big|^2.$$

From the preceding inequality and C_3 we obtain for $i \neq j$

$$\lim_{p\to\infty} \sum_{\alpha=1}^{p} \int_{A_\alpha} | \sigma_{n_i}(x) - \sigma_{n_j}(x) |^2 \, dx = +\infty,$$

and it follows as in the proof of theorem 5.1 that the condition $\mathfrak{M}(\{\sigma_n\}, A) = 0$ fails and necessity is established.

If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number and choose an index p so great that (4.12) and hence (4.13) hold and also so great that

$$(6.11) \qquad \sum_{k=1}^{n} a_{nk} = 1 \quad \text{for} \quad n \geqq p.$$

Choose a constant $R > q$ such that

$$\int_A | s_\mu(x) - s_\nu(x) |^2 < R \quad \text{for} \quad \mu \leqq p,\ \nu \leqq p;$$

then

$$(6.12) \qquad \int_A | s_\mu(x) - s_\nu(x) | \; | s_\xi(x) - s_\eta(x) | \, dx < R \qquad \text{for} \quad \mu \leqq p,\ \nu \leqq p,\ \xi \leqq p,\ \eta \leqq p.$$

If $\mu \geqq p$ and $\nu \leqq p$, then

$$| s_\mu(x) - s_\nu(x) |^2 \leqq 2 \, | s_\mu(x) - s_p(x) |^2 + 2 \, | s_p(x) - s_\nu(x) |^2$$

so that

$$\int_A | s_\mu(x) - s_\nu(x) |^2 \, dx < 2q + 2R < 4R;$$

therefore

$$(6.13) \qquad \int_A | s_\mu(x) - s_\nu(x) | \; | s_\xi(x) - s_\eta(x) | \, dx < 4R \qquad \text{for} \quad \mu \geqq p,\ \xi \geqq p,\ \nu \leqq p,\ \eta \leqq p.$$

Using (6.11) we obtain for $m > p$, $n > p$

$$\sigma_m(x) - \sigma_n(x) = \left(\sum_{\mu=1}^{m} a_{m\mu} s_\mu(x) \right) \left(\sum_{\nu=1}^{n} a_{n\nu} \right) - \left(\sum_{\nu=1}^{n} a_{n\nu} s_\nu(x) \right) \left(\sum_{\mu=1}^{m} a_{m\mu} \right)$$

$$(6.14) \qquad \begin{aligned} &= \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) \\ &= \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) + \sum_{\mu=p+1}^{m} \sum_{\nu=1}^{p} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) \\ &\quad + \sum_{\mu=1}^{p} \sum_{\nu=p+1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) + \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m\mu} a_{n\nu} (s_\mu - s_\nu) \end{aligned}$$

so that

$$(6.15) \qquad \int_A | \sigma_m(x) - \sigma_n(x) |^2 \, dx \leqq \int_A (| \Phi_1 | + | \Phi_2 | + | \Phi_3 | + | \Phi_4 |)^2 \, dx$$

where the Φ's are in order the four terms of the right member of (6.14). We may note that $\Phi_4 \equiv F_5$ and hence, owing to (4.13) and (4.16), write

$$(6.16) \qquad \limsup_{m \to \infty,\ n \to \infty} \int_A | \Phi_4 |^2 \, dx \leqq \limsup_{m \to \infty,\ n \to \infty} (q B_m^2 B_n^2).$$

Employing (6.12), (6.13) and the regularity of (T), the method by which (4.16) was obtained suffices to prove that

$$(6.17) \qquad \limsup_{m \to \infty,\ n \to \infty} \int_A | \Phi_r |^2 \, dx = 0 \qquad (r = 1, 2, 3).$$

Using (6.16), (6.17), and Schwarz's inequality, we find from (6.15) that

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Making use, for the first time, of C_4 we see that $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and sufficiency follows. The same proof establishes theorems 6.2 and 6.3.

THEOREM 6.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable for all* n *and* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n, *and such that* $\mathfrak{M}(\{s_n\}, A) = 0$, C_6 *is necessary and sufficient.*

Necessity is established as in theorem 6.1; the sufficiency proof is a modification of that of theorem 6.1. The same proof establishes theorems 6.5 and 6.6.

7. *Oscillation when* $s_m(x) - s_n(x)$ *is of Summable Square for All Sufficiently Great* m *and* n. The theorems of this section give necessary and sufficient conditions that (T) shall not increase mean square oscillations of the most general sequences of measurable functions for which mean square oscillation is defined.

THEOREM 7.1. *In order that* (T) *may be such that*

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such that* $s_n(x)$ *is measurable for all* n *and* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n, C_4, C_5, *and* C_6 *are necessary and sufficient.*

The necessity of each condition follows at once from a preceding theorem. If $\mathfrak{M}(\{s_n\}, A) = +\infty$, no proof of sufficiency is required. If $\mathfrak{M}(\{s_n\}, A)$ is finite, let q be any greater number. Choose an index p such that $s_m(x) - s_n(x)$ is of summable square for $m \geqq p$, $n \geqq p$, and such that (4.12) and hence (4.13) hold. Due to C_5 and C_6 we can choose an index $N \geqq p$ such that $a_{nk} = 0$, $k = 1, 2, 3, \cdots, p$ for $n > N$ and $\sum_{k=p+1}^{n} a_{nk} = 1$ for $n > N$. Then, referring to (6.15), we see that $\Phi_1 = \Phi_2 = \Phi_3 = 0$ over A, and obtain

$$\mathfrak{M}(\{\sigma_n\}, A) \leqq \limsup_{m\to\infty,\, n\to\infty} \int_A |F_5|^2 dx \leqq \limsup_{m\to\infty,\, n\to\infty} (qB_m^2 B_n^2).$$

Using C_4, we find that $\mathfrak{M}(\{\sigma_n\}, A) \leqq q$ and the theorem follows. The same proof establishes theorems 7.2 and 7.3.

THEOREM 7.4. *In order that* (T) *may be such that* $\mathfrak{M}(\{\sigma_n\}, A) = 0$ *for every sequence* $\{s_n(x)\}$, *defined over a set* A *of measure* $m(A) > 0$, *such

that $s_n(x)$ *is measurable for all* n *and* $\mathfrak{M}(\{s_n\}, A) = 0$, C_5 *and* C_6 *are necessary and sufficient.*

The necessity of each condition follows at once from a preceding theorem; the sufficiency proof is a modification of that of theorem 7. 1. The same proof establishes theorems 7. 5 and 7. 6.

8. *Summary of Results.* Each of the following is an outline of a theorem giving necessary and sufficient conditions that a complex (or real) regular transformation may be such that $\mathfrak{M}(\{\sigma_n\}, A) \leqq \mathfrak{M}(\{s_n\}, A)$ for every complex (or real) sequence $\{s_n(x)\}$ of a stated character defined over a set A of measure $m(A) > 0$.

THEOREM 4. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for all* n; C_4.

THEOREM 4. 2. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for all* n; C_4.

THEOREM 4. 3. *Complex* (T); *Complex* $s_n(x)$, *bounded and measurable for all* n; C_4.

THEOREM 4. 4. *Real* (T); *Real* $s_n(x)$, *bounded and measurable for all* n; C_4.

THEOREM 5. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 5. 2. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 5. 3. *Real* (T); *Real* $s_n(x)$, *bounded above (below) for all* n *and measurable and of summable square for almost all* n; C_4 *and* C_5.

THEOREM 6. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 6. 2. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$, *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 6. 3. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_4 *and* C_6.

THEOREM 7. 1. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

THEOREM 7. 2. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

THEOREM 7.3. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all sufficiently great* m *and* n; C_4, C_5, *and* C_6.

Each of the following is an outline of a theorem giving necessary and sufficient conditions that a complex (or real) regular transformation may be such that $\mathfrak{M}(\{\sigma_n\}, A) = 0$ for every complex (or real) sequence $s_n(x)$ of a stated character, defined over a set A of measure $m(A) > 0$, such that $\mathfrak{M}(\{s_n\}, A) = 0$.

THEOREM 4.5. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for all* n; *none.*

THEOREM 5.4. *Complex* (T); *Complex* $s_n(x)$, *measurable and of summable square for almost all* n; C_5.

THEOREM 5.5. *Real* (T); *Real* $s_n(x)$, *measurable and of summable square for almost all* n; C_5.

THEOREM 5.6. *Real* (T); *Real* $s_n(x)$, *bounded above (below) for all* n *and measurable and of summable square for almost all* n; C_5.

THEOREM 6.4. *Complex* (T); *Complex* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 6.5. *Real* (T); *Real* $s_n(x)$, *measurable for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 6.6. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n *and such that* $s_m(x) - s_n(x)$ *is of summable square for all* m *and* n; C_6.

THEOREM 7.4. *Complex* (T); *Complex* $s_n(x)$ *measurable for all* n; C_5 *and* C_6.

THEOREM 7.5. *Real* (T); *Real* $s_n(x)$, *measurable for all* n; C_5 *and* C_6.

THEOREM 7.6. *Real* (T); *Real* $s_n(x)$, *measurable and bounded above (below) for all* n; C_5 *and* C_6.

9. *Conclusion.* In this paper we have considered in turn the four conditions of which any one might naturally be selected as sufficient for the existence (finite or infinite) of mean square oscillations of sequences. In each case we have found necessary and sufficient conditions that (T) shall not increase mean square oscillations, and that (T) shall preserve convergence in the mean. Thus eight groups of theorems have been obtained. It is a curious fact that these eight groups of theorems should involve as necessary and sufficient conditions the eight possible combinations of none, some or all of the conditions C_4, C_5, and C_6.

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TACTICAL CONFIGURATIONS OF RANK TWO.

By R. D. Carmichael.

1. *Definition and Immediate Examples.* In accordance with the terminology of E. H. Moore * I use the term *tactical configuration of rank two* to denote a combination of l elements into m sets, each set consisting of λ distinct elements and each element occurring in μ distinct sets: it is to be understood that order of sets and order within a set are both immaterial. For such a configuration I use the symbol $\Delta^{\lambda,\mu}_{l,m}$ employed by Coble.† It is obvious that $l\mu = \lambda m$.

In the configurations which we shall usually consider there is a certain symmetry in the rôle played by the l elements and the m sets. We may consider the m sets themselves as elements. The μ sets which contain a given symbol may be thought of as a combination of sets which define that symbol, provided that there is no additional symbol common to these μ sets. From this point of view the configuration gives rise to a new configuration having the symbol $\Delta^{\mu,\lambda}_{m,l}$. The two configurations $\Delta^{\lambda,\mu}_{l,m}$ and $\Delta^{\mu,\lambda}_{m,l}$ are therefore essentially the same when they satisfy the restrictive condition just named. They may be called *associated* configurations.

With each of the m sets of λ elements each we may associate the complementary set of $l - \lambda$ elements, thus forming m sets each of which contains $l - \lambda$ elements; in these sets each element occurs $m - \mu$ times. Thus we have what Coble (*l. c.*, p. 2) calls the configuration *complementary* to $\Delta^{\lambda,\mu}_{l,m}$. Its symbol is $\Delta^{l-\lambda,m-\mu}_{l,m}$.

In the course of the memoir it will become apparent that tactical configurations of the sort defined are important in the theory of permutation groups. Coble (*l. c.*) has found them of essential use in constructing poristic forms in connection with the study of geometrical configurations similar to and including those associated with the Poncelet polygons which arise in the theory of conic sections. He points out (*l. c.*, p. 6) that the same tactical problem appears in the formation of irrational (i. e., non-symmetric) invariants of a set of l points in an $S_{\lambda-1}$ of weight m and degree μ. Hence we see that this tactical problem plays an important role in three widely separated fields each of great interest in itself. To construct such tactical configurations is sometimes a difficult problem. Concerning the difficulty

* *Mathematische Annalen*, Vol. 50 (1898), pp. 226-227 (ftn.).

† *American Journal of Mathematics*, Vol. 43 (1921), pp. 1-19.

Coble says (*l. c.*, p. 6): "Indeed the complications of this tactical problem are the most serious bar to any general discussion of the porisms. Certain series of cases may be treated as a class. . . . As a rule however each case presents its own peculiarities."

Emch * has also recently employed these tactical configurations in connection with several geometric problems.

In this section we shall exhibit a few classes of tactical configurations which are immediately available. In several of the following sections we shall show how to use the finite geometries for the systematic construction of several infinite classes of these configurations. We shall also give a brief account of quadruple systems similar to the usual theory of triple systems. In addition we shall treat briefly certain other remarkable special cases and in particular configurations associated with the Mathieu groups and affording a ready means of constructing these groups.

The finite geometries $PG(k, p^n)$, $k > 1$, furnish at once a certain infinite class of these tactical configurations. The points of the geometry constitute the l elements and the lines of the geometry constitute the m classes. In a similar way from the Euclidean geometry $EG(k, p^n)$, $k > 1$, we may obtain at once certain other configurations of rank two. Since the configurations thus defined are of great importance in the theory of finite groups it is apparent that the general theory of these configurations must have a wide use in this chapter of algebra.

The configuration arising from $EG(k, p^n)$ when $k = 2$ and $p^n = 2$ has the symbol $\Delta^{2,3}_{4,6}$. This particular configuration belongs not only to the infinite class from which it has been taken but also to another; it consists of four things taken in pairs; since the number of pairs is six it follows that all possible pairs appear. In general, one can form $\frac{1}{2}n(n-1)$ pairs from n things, each element occurring in $n-1$ pairs. Thus we have a configuration

$$\Delta^{2,n-1}_{n,n(n-1)/2}.$$

More generally, let us form from n given elements all the sets consisting each of a combination of k distinct elements, k being less than n. Thus we have a configuration $\Delta^{\lambda,\mu}_{l,m}$ where

$$l = n,\ \lambda = k,\ m = \frac{n(n-1)\cdots(n-k+1)}{k!},\ \mu = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!}.$$

* See *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 25-42 and *Journal für Mathematik*, Vol. 162 (1930), pp. 238-255.

We may also form configurations by grouping the points of $PG(k, p^n)$ into the sets formed by the subspaces of a given number s of dimensions where $0 < s < k$. In a similar manner configurations may be formed from $EG(k, p^n)$.

In the case when $p^n = 2$ and $k > 1$ we have in $EG(k, 2)$ the number 2^k of points. Any three points of $EG(k, 2)$ determine a plane, and this plane contains just one additional point of $EG(k, 2)$. Moreover, any three of the points in such a quadruple uniquely determines the quadruple itself. Hence the 2^k points in $EG(k, 2)$ may be taken in 4's, in the way indicated, so that any given triple of these 2^k points occurs in one and just one of the named quadruples. This tactical configuration is therefore a quadruple system. If it is denoted by $\Delta^{\lambda,\mu}_{l,m}$ then we have

$$l = 2^k, \quad \lambda = 4, \quad \mu = (2^k - 1)(2^{k-1} - 1)/3, \quad m = 2^{k-2}(2^k - 1)(2^{k-1} - 1)/3.$$

Thus for $k = 2$ we have just one quadruple—a trivial case. For $k = 3$ we have 14 quadruples of 8 things, each triple occurring just once. For $k = 4$ we have 140 quadruples of 16 things. In the general case, the named quadruple system is left invariant by the collineation group in $EG(k, 2)$ and by no larger permutation group on its elements, as one may show without difficulty.

2. *Dual Configurations.* Owing to the principle of duality in the $PG(k, p^n)$ the principal configurations already formed from it have a certain dual character. It is of interest to construct certain other dual configurations from the special case of $PG(2, p^n)$. From $PG(2, p^n)$ omit a line and all the points on that line, also omit one additional point and all the lines on that point. Then we have left $p^{2n} - 1$ points and $p^{2n} - 1$ lines; moreover, there are p^n retained points on a retained line and also p^n retained lines on a retained point. Considering the points as elements and the lines as sets of elements we are thus led to a configuration $\Delta^{\lambda,\mu}_{l,m}$ where

$$l = m = p^{2n} - 1, \qquad \lambda = \mu = p^n.$$

For $p^n = 2, 3$ these configurations are $\Delta^{2,2}_{3,3}$, $\Delta^{3,3}_{8,8}$. The latter configuration may be represented explicitly by the sets of symbols

136, 147, 158, 238, 245, 267, 357, 468,

where the eight digits are the elements and the triples are those indicated.

This particular configuration was communicated to me orally by A. B. Coble, having been obtained by him in connection with a geometrical investiagtion: it was from an analysis of this particular configuration that I was led to the infinite class of configurations just described.

Let us next consider the configuration obtained from $PG(2, p^n)$ by omitting all the points on a line and all the lines on one point of this line. There remain p^{2n} points and p^{2n} lines; each retained line contains p^n of the retained points, and each retained point is on p^n of the retained lines. Thus we have a $\Delta_{l,m}^{\lambda,\mu}$ with $l = m = p^{2n}$, $\lambda = \mu = p^n$. For $p^n = 3$ we have a $\Delta_{9,9}^{3,3}$ of considerable interest.

Let us now consider the dual configuration formed from $PG(2, p^n)$ in the following manner. We omit all the points on two lines, leaving $p^{2n} - p^n$ points. We also omit all the lines on the common point of these first two lines and also all the lines on one other point of one of these lines. We have thus omitted two lines of points, these two lines having a common point, and also two bundles of lines, these two bundles having a common line. The omitted configuration is dual in character. Hence the points which remain form a set that is dual in character. Grouping these remaining points in collinear sets on the retained lines we have a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = m = p^{2n} - p^n, \qquad \lambda = \mu = p^n - 1.$$

For $p^n = 4$ we have an interesting $\Delta_{12,12}^{3,3}$ which may be represented by the following twelve triples of twelve things:

$$\begin{array}{cccccc} CLO, & CQT, & CMS, & DKT, & DMP, & DOU, \\ EPS, & ELU, & EKQ, & MQU, & KQS, & LPT. \end{array}$$

Let us next omit from $PG(2, p^n)$ three non-collinear points and all the points on the three lines determined by pairs of them and also all the lines on each of these three points. There remain of the $PG(2, p^n)$ the same number of lines and of points, namely, $(p^n - 1)^2$; they fall into sets of $p^n - 2$ each on $p^n - 2$ lines thus giving a configuration $\Delta_{\lambda,\mu}^{l,m}$ with

$$l = m = (p^n - 1)^2, \qquad \lambda = \mu = p^n - 2.$$

The foregoing general configuration may be readily generalized. Let P_r denote a polygon in $PG(2, p^n)$ whose vertices are $A_1, A_2, \cdots, A_r$ and whose sides are A_1A_2, A_2A_3, $A_3A_4, \cdots, A_{r-1}A_r$, A_rA_1. Omit all the points on these r lines and also all the lines on these r vertices. The number of omitted points [omitted lines] is rp^n. Each of the retained lines holds $p^n - r + 1$ of the retained points while each of the retained points is on

the same number of retained lines. We suppose that r is such that $p^n - r + 1$ is an integer s greater than unity and less than $p^n - 1$ (in order to avoid trivial cases). Then we have a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ with

$$\lambda = \mu = s = p^n - r + 1, \qquad l = m = sp^n + 1.$$

Let us now consider the $PG(2, 2^n)$, $n > 2$. Let Q be any complete quadrangle in this plane. Since its diagonal points are collinear it consists of seven points and seven lines. Omitting all the lines on these seven points and all the points on these seven lines, we have from the retained points and lines a $\Delta_{l,m}^{\lambda,\mu}$ for which one readily shows that

$$l = m = 2^{2n} - 6 \cdot 2^n + 8, \qquad \lambda = \mu = 2^n - 6.$$

Several of the configurations which we have obtained from $PG(2, p^n)$ are readily extended to $PG(k, p^n)$ for $k > 1$. Two of these generalizations will now be given.

Let us omit from $PG(k, p^n)$, $k > 1$, one particular $(k-1)$-dimensional subspace $PG(k-1, p^n)$ together with all its points. There remains an $EG(k, p^n)$ containing p^{kn} points. Omit one of these points and each of the $(k-1)$-dimensional subspaces $PG(k-1, p^n)$ which contain this omitted point. The number of $(k-1)$-dimensional subspaces retained is then $p^{kn} - 1$; in each of these we take only those points which are in the named $EG(k, p^n)$. By means of these subspaces we have thus grouped the $p^{kn} - 1$ retained points into $p^{kn} - 1$ sets, each set containing $p^{(k-1)n}$ points and each point appearing in $p^{(k-1)n}$ sets. Thus we are led to a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = m = p^{kn} - 1, \qquad \lambda = \mu = p^{(k-1)n}.$$

In constructing another configuration Δ, let us omit from the $PG(k, p^n)$, $k > 1$, one particular $(k-1)$ space $PG(k-1, p^n)$ together with its points, thus forming an $EG(k, p^n)$ of p^{kn} points. Omit also all $(k-1)$-spaces on a particular one of the points already omitted, retaining the remaining p^{kn} $(k-1)$-spaces. Each of these remaining $(k-1)$-spaces has $p^{(k-1)n}$ points of the $EG(k, p^n)$ on it, while each of these points is on $p^{(k-1)n}$ such spaces. Thus we are led to a dual configuration $\Delta_{l,m}^{\lambda,\mu}$ where $l = m = p^{kn}$, $\lambda = \mu = p^{(k-1)n}$. Of particular interest are the cases $p^n = 2, k = 3$; $p^n = 2, k = 4$; $p^n = 3$; $k = 3$: these lead to configurations with the respective symbols

$$\Delta_{8,8}^{4,4}, \qquad \Delta_{16,16}^{8,8}, \qquad \Delta_{27,27}^{9,9}.$$

It is possible to construct various other dual configurations generalizing several of those here given. In particular, configurations may be constructed in which the elements are lines or other subspaces. But these seem to be of less interest than those already given.

By means of the collineation groups in the finite geometries one may readily determine the permutation groups which are characterized as being the largest permutation groups leaving invariant the configurations described in this section. But the work will not be carried out here.

3. *Other Immediate Examples.* From a cycle $a_1a_2\cdots a_n$ of n elements we may select cyclically the set $a_1a_2\cdots a_k$, $a_2a_3,\cdots a_{k+1},\cdots$, $a_na_1a_2\cdots a_{k-1}$ thus obtaining a configuration $\Delta^{k,k}_{n,n}$.

Let us next take two sets of n things each, say, $a_1, a_2,\cdots, a_n$ and $\alpha_1, \alpha_2,\cdots,\alpha_n$. Each element in one set may be paired with each element in the other set, giving rise to n^2 pairs of the $2n$ elements. Thus we have a $\Delta^{2,n}_{2n,n^2}$. Again, each element in one of the two sets may be paired with every other element in the same set: thus we have a $\Delta^{2,n-1}_{2n,n(n-1)}$. Again, we may make the pairs from each one of the sets run in cyclical order, thus obtaining a configuration with the symbol $\Delta^{2,2}_{2n,2n}$.

The three configurations of the foregoing paragraph are capable of a ready generalization. Generalizing the first of them we have a configuration with the symbol $\Delta^{l,n^{l-1}}_{ln,n^l}$ obtained from l sets of n elements each by forming all the possible combinations of l elements each gotten by taking one element from each of the l sets. One may similarly generalize the other two configurations in the preceding paragraph. Moreover, various other similar configurations are readily formed.

Now let us take l sets of n things each, l being greater than 2. Let these l sets be arranged in cyclical order. Form pairs by taking each element in one set with each element in the set which follows it in cyclical order. Thus we form ln^2 pairs from the ln elements, using each element $2n$ times. This gives rise to a configuration $\Delta^{2,2n}_{ln,ln^2}$, $l > 2$, due to Coble. This configuration is capable of generalization in the following manner. Let us consider l sets of n things each where $l > \lambda$, these l sets being arranged in cyclical order; and let us form combinations of λ elements each, such combinations being formed from λ consecutive sets from the l sets in their fixed cyclical order by taking one element from each of the λ sets in all possible ways. Thus we have ln elements formed into sets of λ elements each, the number of sets being ln^λ and each element appearing in $\lambda n^{\lambda-1}$ sets. This gives rise to a configuration

$$\Delta^{\lambda,\lambda n\lambda-1}_{ln,ln\lambda}, \qquad l > \lambda.$$

For the case in which $l = n + 1$ and $\lambda = n$ we have $\Delta^{n,\quad n^n}_{n(n+1),n^n(n+1)}$. For $n = 2$ this becomes $\Delta^{2,4}_{6,12}$; and this configuration consists of all the pairs of the six elements involved except the three pairs from which the configuration was formed.

Other configurations are readily formed by various modifications of the methods employed in this section.

4. *Configurations Associated with Coble's Box Porism.* In the 3-space $PG(3, p^n)$ there are $p^n + 1$ points on a line l and $p^n + 1$ planes on the same line. Let us take p^n of these planes and a point P not on any of these p^n planes (and hence on the remaining plane through l). Let Q be any point on l. In addition to the p^n planes already retained, keep also the p^{2n} planes which are not on the line PQ. We thus retain $p^{2n} + p^n$ planes. Retain the p^{3n} points which are not on the plane through P and l; these points form an $EG(3, p^n)$. The points retained in a given one of the p^n planes first selected and the lines in which that plane is cut by the retained planes on P form a configuration

$$\Delta^{p^n,p^n}_{p^{2n},p^{2n}}.$$

Hence the p^{3n} retained points appear in sets of p^n each on the $p^{2n} + p^n$ retained planes; moreover, each of the retained points appears on one of the retained planes through l and on just p^n of the retained planes on P. We are thus led to a configuration having the symbol

$$\Delta^{p^{2n},p^n+1}_{p^{3n},p^{2n}+p^n}.$$

When $p^n = 2$ we have here a configuration with the symbol $\Delta^{4,3}_{8,6}$. It is based on the $PG(3, 2)$. It may be shown that this leads to a configuration equivalent to that defined by the following scheme:

DEFG, LMNO, DELM, FGNO, DGLO, EFMN.

If these six quadruples in the order written are numbered 1, 2, · · ·, 6, then the eight letters named in them are determined by triples of digits according to the following correspondence:

135	136	145	146	235	236	245	246
D	*E*	*G*	*F*	*L*	*M*	*O*	*N*

These eight triples of six elements form the configuration $\Delta_{6,8}^{8,4}$ belonging to the box porism of Coble (*l. c.*, p. 15). The latter is therefore exhibited as belonging to an infinite class of configurations: the class was suggested by this example.

Another infinite class of configurations having the same symbols as the foregoing may be constructed in the following manner. From $PG(3, p^n)$ form the corresponding $EG(3, p^n)$ by omitting a plane with its points. Let P be a point on this plane; then there are $p^{2n} + p^n$ additional planes on P; these are to be retained. The p^{3n} points of the $EG(3, p^n)$ fall on these planes, p^{2n} points on each plane thus considered. Moreover a given one of these points is on each of the planes containing the line joining this point to P, and hence it is on $p^n + 1$ of the retained planes. We are thus led to another general configuration with the same symbol as that which appears in the preceding paragraph. These configurations, however, have a certain degenerate character so that they do not give rise to associated configurations by means of the method just employed in the preceding paragraph.

5. *Certain Additional Configurations.* From the $PG(k, p^n)$, $k > 2$, let us omit a line of points and also all the lines on each of these points. The number of points remaining is l where

$$l = p^{2n} + p^{3n} + \cdots + p^{kn}.$$

By computing the number of omitted lines it is readily shown that the number m of retained lines is

$$m = \frac{(p^{(k+1)n} - 1)(p^{kn} - 1)}{(p^{2n} - 1)(p^n - 1)} - (1 + p^n)(p^n + p^{2n} + \cdots + p^{(k-1)n}) - 1.$$

The retained points fall λ at a time on the retained lines, where $\lambda = 1 + p^n$, each point appearing on μ of the lines, where

$$\mu = p^{2n} + p^{3n} + \cdots + p^{(k-1)n}.$$

This gives rise to a $\Delta_{l,m}^{\lambda,\mu}$ where l, m, λ, μ have the values given.

To form another configuration let us omit from $PG(3, p^n)$ the points on two non-intersecting lines and all the lines through these points. This leaves $p^n(p^{2n} - 1)(p^n - 1)$ lines of the $PG(3, p^n)$. Each of these contains $p^n + 1$ points of the $PG(3, p^n)$; and each of these retained points is on $p^{2n} - p^n$ lines. Hence the retained points and lines yield a configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l=(p^n+1)(p^{2n}-1), \qquad m=p^n(p^n-1)(p^{2n}-1),$$
$$\lambda=p^n+1, \qquad \mu=p^n(p^n-1).$$

Again, from the $PG(2k+1, p^n)$ let us omit the points of a k-dimensional subspace S_k and also all the $(2k)$-spaces containing S_k. We thus retain l points and l $(2k)$-spaces where

$$l=p^{(2k+1)n}+p^{2kn}+\cdots+p^{(k+1)n}.$$

Each of the retained $(2k)$-spaces has λ retained points where

$$\lambda=p^{2kn}+p^{(2k-1)n}+\cdots+p^{kn},$$

and each of the retained points lies on λ of the retained $(2k)$-spaces. Thus we are led to a configuration $\Delta_{l,l}^{\lambda,\lambda}$ where λ and l have the values just given.

6. *Subgeometries and the Complementary Sets.* Let ν be any proper factor of n. Then in $PG(k, p^n)$ there is included the geometry $PG(k, p^\nu)$, namely, those points of $PG(k, p^n)$ whose coördinates may be taken as marks of the $GF[p^\nu]$ included in the $GF[p^n]$. We shall denote by $C(k, p^n, p^\nu)$ the complementary set of points, namely, the points of $PG(k, p^n)$ which are not contained in the included $PG(k, p^\nu)$. The number l of points in $C(k, p^n, p^\nu)$ is

$$l=(p^{kn}-p^{k\nu})+(p^{(k-1)n}-p^{(k-1)\nu})+\cdots+(p^n-p^\nu).$$

If a line in $PG(k, p^n)$ contains two points of $PG(k, p^\nu)$ it contains all the points of a line in $PG(k, p^\nu)$. Hence the lines of $PG(k, p^n)$ may be separated into three classes: the first class consists of those lines each of which contains a whole line of the $PG(k, p^\nu)$; the second class consists of those lines each of which contains just one point of the $PG(k, p^\nu)$; the third class consists of those lines containing no point of the $PG(k, p^\nu)$. The numbers of lines in these three classes are readily shown to be respectively

$$\frac{(p^{(k+1)\nu}-1)(p^{k\nu}-1)}{(p^{2\nu}-1)(p^\nu-1)}, \quad \left(\frac{p^{kn}-1}{p^n-1}-\frac{p^{k\nu}-1}{p^\nu-1}\right)\frac{p^{(k+1)\nu}-1}{p^\nu-1},$$
$$\frac{(p^{(k+1)n}-1)(p^{kn}-1)}{(p^{2n}-1)(p^n-1)}-\frac{p^{(k+1)\nu}-1}{p^\nu-1}\left(\frac{p^{kn}-1}{p^n-1}-\frac{p^\nu(p^{k\nu}-1)}{p^{2\nu}-1}\right).$$

It is not difficult to show that the third class is the null class when and only when $k=2$ and $n=2\nu$.

With these classes we readily construct tactical configurations as follows.

Let us consider the second class of lines in the case when $k = 2$ and $n = 2\nu$. Each of the $p^{2\nu} + p^\nu + 1$ lines of the $PG(2, p^\nu)$, when extended to a line of $PG(2, p^{2\nu})$, contains just $p^{2\nu} - p^\nu$ points of $C(2, p^{2\nu}, p^\nu)$; and no point P of $C(2, p^{2\nu}, p^\nu)$ occurs on two such extended lines. Hence each of the $(p^{2\nu} - p^\nu)(p^{2\nu} + p^\nu + 1)$ points of $C(2, p^{2\nu}, p^\nu)$ occurs on one and just one line which contains a line of $PG(2, p^\nu)$. Hence each point P of $C(2, p^{2\nu}, p^\nu)$ lies on just $p^{2\nu}$ lines of the second class, this being the number of lines joining P to points of $PG(2, p^\nu)$ other than the line of $PG(2, p^\nu)$ on the extension of which P lies. Moreover, each line of the second class contains just $p^{2\nu}$ points of $C(2, p^{2\nu}, p^\nu)$. Hence the $(p^{2\nu} - p^\nu)(p^{2\nu} + p^\nu + 1)$ points of $C(2, p^{2\nu}, p^\nu)$ lie $p^{2\nu}$ at a time on the $(p^{2\nu} - p^\nu)(p^{2\nu} + p^\nu + 1)$ lines of the second class and each point is on just $p^{2\nu}$ of these lines. This gives rise to a tactical configuration $\Delta_{l,m}^{\lambda,\mu}$, where

$$\lambda = \mu = p^{2\nu}, \qquad l = m = (p^{2\nu} - p^\nu)(p^{2\nu} + p^\nu + 1).$$

In the case when $p^\nu = 2$ this gives a $\Delta_{14,14}^{4,4}$; this can be represented explicitly in the following form in which the fourteen columns denote the fourteen sets of four points each (each point occurring in four sets):

E	*D*	*I*	*H*	*E*	*D*	*R*	*D*	*I*	*E*	*N*	*E*	*D*	*H*
M	*L*	*M*	*L*	*I*	*H*	*S*	*M*	*L*	*H*	*O*	*L*	*I*	*M*
O	*Q*	*Q*	*P*	*P*	*O*	*T*	*P*	*O*	*Q*	*P*	*N*	*N*	*N*
T	*S*	*U*	*T*	*S*	*U*	*U*	*R*	*R*	*R*	*Q*	*U*	*T*	*S*

From the last foregoing general configuration a certain reduced configuration is readily obtained. Let us omit from $PG(2, p^\nu)$ one of its lines and at the same time omit from $PG(2, p^{2\nu})$ the line L which has $p^\nu + 1$ points in common with the omitted line $PG(2, p^\nu)$. This line contains $p^{2\nu} - p^\nu$ points of $C(2, p^{2\nu}, p^\nu)$. The remaining points of $C(2, p^{2\nu}, p^\nu)$ are $(p^{2\nu} - p^\nu)(p^{2\nu} + p^\nu)$ in number. These points fall $p^{2\nu}$ at a time on those lines of the second class other than the lines containing each a point of L which is in the set $C(2, p^{2\nu}, p^\nu)$. These latter lines are $(p^{2\nu} - p^\nu)p^{2\nu}$ in number, since each of the excluded $p^{2\nu} - p^\nu$ points is on just $p^{2\nu}$ lines of the second class and no two of them are on the same line of the second class. Excluding these lines and retaining the others of the second class we have $p^\nu(p^{2\nu} - 1)$ retained lines. Each of the retained points is on just p^ν of the retained lines. Hence we have a tactical configuration $\Delta_{l,m}^{\lambda,\mu}$ where

$$l = p^{2\nu}(p^{2\nu} - 1), \quad \lambda = p^{2\nu}, \quad \mu = p^\nu, \quad m = p^\nu(p^{2\nu} - 1).$$

For $p^\nu = 2$ this is a $\Delta_{12,6}^{4,2}$. The associated configuration $\Delta_{6,12}^{2,4}$ has an obvious generalization to a configuration

$$\Delta_{2n,2n(n-1)}^{2,\ 2(n-1)}$$

consisting of all the pairs of the $2n$ symbols $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ except the pairs α_1, α_2; α_3, α_4; $\cdots$; $\alpha_{2n-1}, \alpha_{2n}$, each α occurring in $2(n-1)$ pairs. And this in turn is capable of an immediate generalization to the case of kn things taken k at a time except for the omission of n sets of k each, the latter sets involving each symbol once and just once.

Let us consider the second class of lines in the case when $k = 2$ and $n = \rho\nu$, $\rho > 2$. The number l of points in $C(2, p^{\rho\nu}, p^\nu)$ and the number m of lines in the second class and the number N of lines in the third class are now respectively:

$$l = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} + p^\nu + 1), \qquad m = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1),$$
$$N = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}).$$

Moreover, each line of the second class contains just $p^{\rho\nu}$ points of $C(2, p^{\rho\nu}, p^\nu)$. We may separate the points of $C(2, p^{\rho\nu}, p^\nu)$ into two subclasses $C_1(2, p^{\rho\nu}, p^\nu)$ and $C_2(2, p^{\rho\nu}, p^\nu)$, those of the subclass C_1 being each on a line of $PG(2, p^{\rho\nu})$ which contains $p^\nu + 1$ points of the $PG(2, p^\nu)$ while the subclass C_2 consists of the remaining points of C. Now these subclasses C_1 and C_2 contain l_1 and l_2 points respectively where

$$l_1 = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1), \qquad l_2 = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}),$$

a result which may be proved as follows. The $PG(2, p^\nu)$ contains $p^{2\nu} + p^\nu + 1$ lines and each of these lines has $p^{\rho\nu} - p^\nu$ points of C_1 while no point of C_1 is on two of these lines, since two such lines have a point of $PG(2, p^\nu)$ in common. Hence l_1 has the value just given; then l_2 is obtained from the formula $l_2 = l - l_1$.

Each point of C_1 is on just $p^{2\nu}$ lines of the second class, since it is on just one line of the first class and this line contains just $p^\nu + 1$ of the $p^{2\nu} + p^\nu + 1$ points of the $PG(2, p^\nu)$; and each point of C_2 is on just $p^{2\nu} + p^\nu + 1$ lines of the second class. But just $p^{\rho\nu} + 1$ lines of $PG(2, p^{\rho\nu})$ pass through any given point of this geometry. Hence each point of C_1 is on just $p^{2\nu} - p^\nu$ lines of the third class, and each point of C_2 is on just $p^{\rho\nu} - p^{2\nu} - p^\nu$ lines of the third class. Every line of the second class contains just as many points of C_1 as there are lines in $PG(2, p^\nu)$ not containing the point which this line of the second class has in common with

$PG(2, p^\nu)$, and this number is $p^{2\nu}$; therefore every line of the second class contains just $p^{\rho\nu} - p^{2\nu}$ points of C_2.

Now we have seen that the l_1 points of C_1 fall, in sets of $p^{2\nu}$ each, on the m lines of the second class, each point of C_1 belonging to just $p^{2\nu}$ lines of the second class. Thus we have a $\Delta^{\lambda,\mu}_{l,m}$ with

$$l = m = (p^{\rho\nu} - p^\nu)(p^{2\nu} + p^\nu + 1), \qquad \lambda = \mu = p^{2\nu}.$$

For $\rho = 3$ and $p^\nu = 2$ we have thus a $\Delta^{4,4}_{42,42}$.

Again, the l_2 points of C_2 fall, in sets of $p^{\rho\nu} - p^{2\nu}$ each, on the m lines of the second class, each point of C_2 belonging to just $p^{2\nu} + p^\nu + 1$ lines of the second class. Thus we have a $\Delta^{\lambda,\mu}_{l,m}$ where

$$\lambda = p^{\rho\nu} - p^{2\nu}, \qquad \mu = p^{2\nu} + p^\nu + 1, \qquad l = l_2,$$

and where m and l_2 have the values already given.

Each line of the third class contains just $p^{2\nu} + p^\nu + 1$ points of C_1 since it contains no point of $PG(2, p^\nu)$ and has one and just one point in common with each of the $p^{2\nu} + p^\nu + 1$ lines each of which contains $p^\nu + 1$ points of $PG(2, p^\nu)$. Hence each line of the third class contains also $p^{\rho\nu} - p^{2\nu} - p^\nu$ points of C_2.

From the foregoing results we see that the l_1 points of C_1 fall, in sets of $p^{2\nu} + p^\nu + 1$ each, on the N lines of the third class. This defines a tactical configuration of rank two.

Similarly, we see that the l_2 points of C_2 fall, in sets of $p^{\rho\nu} - p^{2\nu} - p^\nu$ each, on the N lines of the third class, each point of C_2 belonging to just $p^{\rho\nu} - p^{2\nu} - p^\nu$ lines of the third class. Thus we have a $\Delta^{\lambda,\lambda}_{l,l}$, where

$$\lambda = p^{\rho\nu} - p^{2\nu} - p^\nu, \qquad l = (p^{\rho\nu} - p^\nu)(p^{\rho\nu} - p^{2\nu}).$$

It is evident that other configurations may readily be constructed by means of finite geometries of more than two dimensions and the subgeometries contained within them.

7. *Quadruple Systems.* If n elements $x_1, x_2, \cdots, x_n$ can be arranged in quadruples so that each triple $x_\alpha x_\beta x_\gamma$ of distinct elements occurs in one and just one quadruple, then the arrangement so made is called a *quadruple system.* The number n of elements in a quadruple system must be of one of the forms $6m + 2$ and $6m + 4$, as one may readily prove by showing that each of the numbers

$$n(n-1)(n-2)/4\cdot 3\cdot 2, \quad (n-1)(n-2)/3\cdot 2, \quad (n-2)/2,$$

must be an integer: the first of these numbers is the number of quadruples in the system; the second is the number of quadruples containing a given element; while the third is the number of quadruples containing a given pair of elements. The quadruples containing a given element evidently lead to a triple system on the remaining elements.

From a given quadruple system on the n elements $x_1, x_2, \cdots, x_n$ one may form a quadruple system on the $2n$ elements $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n'$ in the following manner. For each quadruple $x_\alpha x_\beta x_\gamma x_\delta$ of the given set form also the quadruple $x_\alpha' x_\beta' x_\gamma' x_\delta'$ and retain $x_\alpha x_\beta x_\gamma x_\delta$. For each quadruple containing the given pair $x_\alpha x_\beta$, as for instance $x_\alpha x_\beta x_\lambda x_\mu$, form also the quadruple $x_\alpha' x_\beta' x_\lambda x_\mu$. Form also the quadruples $x_\alpha x_\beta x_\alpha' x_\beta'$ for every pair (α, β) of the set $1, 2, \cdots, n$. The total number of quadruples thus formed is

$$2(n)(n-1)(n-2)/24 + n(n-1)(n-2)/4 + n(n-1)/2,$$

or

$$2n(2n-1)(2n-2)/24.$$

This is just the required number of quadruples for a quadruple system of $2n$ elements. Therefore the named quadruples form a quadruple system provided that no triple occurs in two quadruples. That this condition is met is readily shown by considering the triples of each of the forms $x_\rho x_\sigma x_\tau$, $x_\rho' x_\sigma' x_\tau'$, $x_\rho x_\sigma x_\tau'$, $x_\rho' x_\sigma' x_\tau$. Hence from a given quadruple system on n elements one may construct (in the manner indicated) a quadruple system on $2n$ elements.

Now $x_1 x_2 x_3 x_4$ forms a (trivial) quadruple system. Applying to it the method of the previous paragraph one obtains a quadruple system on eight elements; and it is easy to show that this is the only quadruple system on eight elements. From the quadruple system on eight elements one may form one on 16 elements; from this, one on 32 elements; and so on. Thus one has quadruple systems on 2^k elements for $k=3, 4, 5 \cdots$. These are the same as the quadruple systems formed at the end of § 1 by means of the finite geometries.

Now consider the collineation group $C(1, 3^k)$ of the $PG(1, 3^k)$. It has a subgroup consisting of those transformations

$$x' = (\alpha x + \beta)/(\gamma x + \delta),$$

for which α, β, γ, δ belong to the Galois field $GF[3]$; this subgroup is of order $4\cdot 3\cdot 2$; it permutes among themselves the elements ∞, 0, 1, 2; these

elements are left individually fixed by the transformation $x' = x^3$; this transformation and the group of order 24 just mentioned generate a group of order $24k$, each element of which leaves fixed the set ∞, 0, 1, 2. Hence the group $C(1, 3^k)$ transforms this quadruple into $(3^k + 1)3^k(3^k - 1)/24$ quadruples [as does also the projective group $P(1, 3^k)$ of $PG(1, 3^k)$]. Since the group is triply transitive it follows that every triple of the $3^k + 1$ points of $PG(1, 3^k)$ occurs among these quadruples. The quadruples therefore constitute a quadruple system. When $k = 1$ we have a trivial case. When $k = 2$ we have a quadruple system on 10 elements.

From the three preceding paragraphs it follows that quadruple systems of n elements certainly exist for every number n of the form

$$n = (3^k + 1)2^l, \qquad (k = 1, 2, 3, \cdots, \; l = 0, 1, 2, \cdots).$$

The general problem of the existence of quadruple systems of n elements when n is of the form $6m + 2$ or $6m + 4$ appears not to have been solved.

Let us return to the quadruple system on $3^k + 1$ elements already constructed. Those quadruples which contain the element ∞ lead to a triple system on the 3^k elements exclusive of ∞. It may be shown that this is the same as the triple system afforded by the lines of $EG(k, 3)$. Its group is therefore the projective group $EP(k, 3)$, a doubly transitive group of degree 3^k and order

$$3^k(3^k - 1)(3^k - 3)(3^k - 3^2) \cdots (3^k - 3^{k-1}).$$

This triple system may also be constructed (in a manner now obvious) by means of the transformation group $x' = \alpha x + \beta$ in the $GF[3^k]$; and when so constructed it leads at once to the larger doubly transitive group just named—a good example of the way in which configurations often lead from a given multiply transitive group to a larger one containing it.

8. *Configurations Associated with the Mathieu Groups.* The Mathieu groups of degrees 11, 12, 22, 23, 24 (one of each degree) are remarkable for two things: (a) they seem to be the only known simple groups which do not appear among the known infinite classes of simple groups; (b) among them are found the only known four-fold and five-fold transitive groups other than the alternating and symmetric groups. Examples which stand apart in such a way possess a peculiar interest on account of their isolation. It therefore seems worth while to present (without any details) a very direct method for constructing these groups by means of configurations and to indicate some

of their properties which are made manifest by means of these configurations.

The linear fractional group modulo 11 of order $12 \cdot 11 \cdot 5$ is often represented as a doubly transitive group of degree 12 on the symbols ∞, 0, 1, 2, $\cdots$, 10. From the twelve symbols which this transitive group permutes one may select a set of six, namely, ∞, 1, 3, 4, 5, 9, such that the set is transformed into itself by just five elements of this group. The whole group therefore permutes the set of six symbols into 132 such sets. If any five symbols are selected from the twelve they appear in one and just one of these sextuples. The 132 sextuples therefore afford an interesting configuration on 12 symbols which may well be called a sextuple system, in analogy with the terminology employed in the preceding section. The symbol ∞ appears in just 66 of these sextuples, whence it follows readily that these 66 sextuples afford a configuration of 66 quintuples on the set 0, 1, 2, $\cdots$, 10. These may be said to form a quintuple system since each set of four of the symbols appears in one and just one of the quintuples. Any one of the 11 elements occurs in just 30 quintuples from which a quadruple system on 10 elements may be formed by omitting that element. From this in turn the triple system on nine elements may be constructed.

If one seeks the largest permutation group G on the twelve symbols, each element of which leaves invariant the named sextuple system, it is found that G is a five-fold transitive group of degree 12 and order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$. This is the Mathieu group of degree 12. Its largest subgroup, each element of which leaves one given symbol fixed, is the Mathieu group of degree 11, a fourfold transitive group of order $11 \cdot 10 \cdot 9 \cdot 8$. Moreover it is the group belonging to the quintuple system already named.

From the foregoing considerations it follows also that the Mathieu group of degree 12 contains a subgroup of order $10 \cdot 9 \cdot 8$ each element of which leaves fixed a given one of the 132 sextuples. This subgroup is intransitive, having two transitive constituents each of degree 6. It thus sets up a simple isomorphism of the symmetric group of degree 6 with itself; and the isomorphism so established is an outer isomorphism. This outer isomorphism is therefore an essential element in the structure of the Mathieu group of degree 12.

The linear fractional group modulo 23 of order $24 \cdot 23 \cdot 11$ is often represented as a doubly transitive group of degree 24 on the symbols ∞, 0, 1, 2, $\cdots$, 22. This transitive group contains a subgroup of order 8 each element of which transforms into itself the set ∞, 0, 1, 3, 12, 15, 21, 22 of eight elements, while the whole group transforms this set into $3 \cdot 23 \cdot 11$ sets of eight each. This configuration of octuples has the remarkable property

that any given set of five of the 24 symbols occurs in one and just one of these octuples. The largest permutation group Γ on the 24 symbols, each element of which leaves this configuration invariant, is a five-fold transitive group of degree 24 and order $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$. This is the Mathieu group of degree 24. Its four-fold and three-fold transitive subgroups of degrees 23 and 22 are the Mathieu groups of these degrees. With these two subgroups respectively we may associate (in a manner now obvious) configurations on 23 and 22 letters respectively. The former consists of septuples such that any set of four of the 23 elements occurs in one and just one septuple; the latter consists of sextuples such that any set of three of the 22 elements in it occurs in one and just one sextuple.

The latter set of sextuples on 22 symbols leads readily to 21 quintuples on 21 symbols; it may be shown that these quintuples constitute the lines of the geometry $PG(2, 2^2)$ of 21 points.

The Mathieu group of degree 24 contains a subgroup of index $3 \cdot 23 \cdot 11$ each element of which leaves invariant a given octuple of the previously named configuration of octuples. This subgroup permutes the eight symbols in this octuple according to the alternating group of degree 8; it permutes the remaining 16 symbols according to a triply transitive group of degree 16 and order $16 \cdot 15 \cdot 14 \cdot 12 \cdot 8$; the latter of these two groups is $(16, 1)$ isomorphic with the former. This isomorphism is essential in the structure of the Mathieu group of degree 24. By means of this isomorphism and the known lists of groups of degree not exceeding 8 it is easy to find all the primitive groups of degree 16 contained in the named triply transitive group of degree 16: it turns out that they are 20 in number: these are all the primitive groups of degree 16 except the alternating and symmetric groups of this degree (Miller, *American Journal of Mathematics,* Vol. 20 (1899), pp. 229-241). By means of the named $(16, 1)$ isomorphism it may also be shown without much difficulty that for every transitive group of degree 5 there exists a doubly transitive group of degree 16 which is $(48, 1)$ isomorphic with the group of degree 5.

9. *Configurations of Marks in* $GF[p^n]$. Let ω be a primitive mark of the Galois field $GF[p^n]$ and let μ be any (positive) factor of $p^n - 1$. Write $p^n - 1 = \mu k$. Let G_k, $G_k \equiv G_k[p^n]$, be the group, of order μp^n, consisting of the transformations $x' = \alpha x + \beta$, where β runs over all the marks of $GF[p^n]$ and α over the k-th power marks of this field. The set of marks $1, \omega^k, \omega^{2k}, \cdots, \omega^{(\mu-1)k}$ is left invariant as a set by the transformations $x' = \alpha x$ of G_k and by no other transformations of G_k, since μ is prime to p, while the transformations $x' = x + \beta$ of G_k permute this set into the p^n (distinct)

sets in the following columns:

$$\begin{array}{ccccc}
1 & 1+\omega & 1+\omega^2 & \cdots & 1+\omega^{p^n-1} \\
\omega^k & \omega^k+\omega & \omega^k+\omega^2 & \cdots & \omega^k+\omega^{p^n-1} \\
\omega^{2k} & \omega^{2k}+\omega & \omega^{2k}+\omega^2 & \cdots & \omega^{2k}+\omega^{p^n-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\omega^{(\mu-1)k} & \omega^{(\mu-1)k}+\omega & \omega^{(\mu-1)k}+\omega^2 & \cdots & \omega^{(\mu-1)k}+\omega^{p^n-1}.
\end{array}$$

Hence the configuration A_k, $A_k \equiv A_k[p^n]$, on the marks of $GF[p^n]$, defined by this array is invariant under G_k.

Let Γ_k, $\Gamma_k \equiv \Gamma_k[p^n]$, be the largest transformation group on the marks of $GF[p^n]$ each element of which leaves invariant the configuration A_k. Then Γ_k contains G_k as a subgroup. It is easy to show that G_k is the largest subgroup of Γ_k that is contained in the group of all linear transformations in $GF[p^n]$. The groups G_k and Γ_k induce permutation groups G_k and Γ_k respectively on the p^n marks of $GF[p^n]$; it is sometimes more convenient to deal with these permutation groups than with the transformation groups by means of which they are defined.

Let us consider the case when each pair of marks occurs in ν and just ν of the sets in $A_k[p^n]$. Then by a count of pairs it is seen to be necessary that $\mu = k\nu + 1$ and hence that $p^n = 1 + k + \nu k^2$. Since 0 and 1 must occur together in just ν of the sets it follows that the $GF[p^n]$ must have the property that there are just ν pairs of k-th powers in $GF[p^n]$ such that the elements of each pair differ by 1. The configurations thus arising would doubtless reward further investigation, especially the case when $\nu = 2$ and $p^n = 1 + k + 2k^2$.

We consider further the general case when $k = 2$. Then $p^n = 4\nu + 3$ and G_2 contains a transformation replacing the pair $0, 1$ by any preassigned pair. Hence any preassigned pair of marks occurs in the same number of sets of $A_2[p^n]$ as any other pair. Thus we have the theorem:

If $p^n = 4\nu + 3$ then each pair of marks occurs in ν and just ν sets belonging to the configuration $A_2[p^n]$; there are just ν pairs of squares in $GF[p^n]$ such that the elements of each pair differ by 1; any two sets in $A_2[p^n]$ have just ν elements in common.

It is easy to show that $A_2[7]$ is the same as $PG(2,2)$ and hence that the group belonging to it is the doubly transitive group of degree 7 and order $7 \cdot 6 \cdot 4$.

The $A_2[11]$ is an interesting configuration consisting of the 11 quintuples into which the set 1, 4, 5, 9, 3 is changed by the cyclic permutation $(0, 1, 2, \cdots, 10)$. The group belonging to it is the doubly transitive group

of degree 11 and order $11 \cdot 10 \cdot 6$. A given pair of symbols occurs in just two quintuples of $A_2[11]$. Furthermore, any two of these quintuples have just two symbols in common. With each of the 55 pairs of these quintuples we may associate the quintuple formed by taking the two symbols common to the pair and the three symbols absent from both pairs; thus we have 55 additional quintuples. If we adjoin them to the 11 quintuples in $A_2[11]$ we have 66 quintuples forming the quintuple system described in § 8. The group belonging to it is of order $11 \cdot 10 \cdot 9 \cdot 8$, as we have seen, and hence is much larger than the group of order $11 \cdot 10 \cdot 6$ which belongs to each of the parts from which we have formed the quintuple system.

In general, when $p^n = 4\nu + 3$, any two sets in $A_2[p^n]$ have just ν elements in common; these and the $\nu + 1$ elements absent from both pairs form a set of $2\nu + 1$ elements; thus we have a configuration $B_2[p^n]$ consisting of $\frac{1}{2}p^n(p^n - 1)$ sets of $2\nu + 1$ elements each; combining $A_2[p^n]$ and $B_2[p^n]$ we have a configuration $C_2[p^n]$ containing the quintuple system of the preceding paragraph as a special case. These configurations $C_2[p^n]$ would doubtless reward further investigation.

Let us now consider the group G of transformations of the form

$$t' = (\alpha t + \beta)/(\gamma t + \delta), \qquad \alpha\delta - \beta\gamma = \text{square},$$

in the Galois field $GF[p^n]$ where p is an odd prime. Let S denote the set of $\frac{1}{2}(p^n + 1)$ elements consisting of ∞ and the square marks of $GF[p^n]$ and denote by $D[p^n]$ the configuration consisting of the sets into which S is transformed by G. I have not developed a theory of the configurations $D[p^n]$ though they seem to be of considerable interest. The configuration $D[7]$ is the quadruple system on eight elements; the group belonging to it is the triply transitive group of degree 8 and order $8 \cdot 7 \cdot 6 \cdot 4$. The $D[11]$ is the sextuple system on 12 elements which (§ 8) characterizes the Mathieu five-fold transitive group of degree 12.

10. *Configurations Associated with Multiply Transitive Groups.* Let G be a multiply transitive group of degree n whose degree of transitivity is k; and let G have the property that a set of m elements $(k < m < n)$ exists in G such that, when k of these m elements are changed by a permutation of G into k of these m elements, then all the m elements are permuted among themselves. Then the largest subgroup of G which permutes these m elements among themselves permutes them according to a transitive group which is at least k-fold transitive. Moreover, of $k > \frac{1}{3}m + 1$ (and also under certain other conditions) it follows (from the theory of multiply transitive groups)

that these m elements are then permuted by the named subgroup according to the alternating or the symmetric group of degree m; and it is certainly the symmetric group when $m = k + 1$.

Denote the order of G by $n(n-1)\cdots(n-k+1)\lambda$ and let G_1 be the subgroup of G of order λ leaving fixed each of a given set of k elements. Let H be the largest subgroup of G which permutes the named m elements among themselves and denote by $m(m-1)\cdots(m-k+1)\mu$ the order of the group Γ by which these m elements are permuted by H. Then μ is the order of the largest subgroup Γ_1 of Γ which leaves fixed each of k given elements; hence Γ_1 is a subgroup of the group induced by G_1 on the m elements on which Γ operates: therefore μ is a factor of λ.

Let ρ be the order of the largest subgroup K of G which leaves fixed each of the given m elements and let σ be the order of the largest subgroup L of G which leaves fixed each symbol of G not in the set of m given elements. Then G_1 is (ρ, σ) isomorphic with Γ_1. Hence $\lambda\sigma = \rho\mu$ and the order of H is $m(m-1)\cdots(m-k+1)\lambda\sigma$.

Thence it follows that G permutes the given m elements into

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1)\sigma$$

sets of m elements each, thus forming a configuration which we denote by E. Since G is k-fold transitive it follows that a given set of k elements occurs in just as many of these sets of m each as any other set of k elements. Since each set of k elements must appear at least once it follows that the number of sets must be at least as large as

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1).$$

Therefore $\sigma = 1$ and each set of k elements appears in one and just one set of m elements in the configuration E.

Thus we have the following theorem:

Let G be a multiply transitive group of degree n whose degree of transitivity is k; and let G have the property that a set S of m elements exists in G $(k < m < n)$ such that when k of these elements S are changed by a permutation of G into k of these elements then all these m elements are permuted among themselves. Then the identity is the only element in G which leaves fixed each of the $n - m$ elements not in S; G permutes the m elements of S into

$$n(n-1)\cdots(n-k+1)/m(m-1)\cdots(m-k+1)$$

sets of m elements each, thus forming a configuration E having the property that any (whatever) set of k elements appears in one and just one of the sets which constitute E.

It is clear that a necessary condition for meeting the hypotheses of the theorem is that k, m, n shall be such that each of the numbers

$$\frac{n-k+1}{m-k+1}, \frac{(n-k+2)(n-k+1)}{(m-k+2)(m-k+1)}, \cdots, \frac{n(n-1)\cdots(n-k+1)}{m(m-1)\cdots(m-k+1)}$$

shall be an integer. If $m=6$ and $k=5$ it may then be readily shown that n is of one of the forms 6ρ or $6\rho+10$ where ρ is not divisible by 5. In the case $n=12$ we have already had an illustration of the theorem afforded by the Mathieu five-fold transitive group of degree 12. If $m=8$ and $k=5$ it may be shown that n must be of one of the forms $20\mu+4$ and $20\mu+8$ and that it must also be congruent modulo 7 to 1, 2, 3 or 4. The smallest values of n ($n>8$) meeting these conditions are 24, 44, 88, 108. The Mathieu five-fold transitive group of degree 24 affords an illustration of the theorem for the case $n=24$, $m=8$, $k=5$, as may be seen from § 8.

The finite geometries $PG(\lambda, p^\nu)$ afford examples of the configurations defined in the foregoing theorem, k being 2 and m being the number of points on a line. Similar examples arise also from the $EG(\lambda, p^\nu)$. In these cases one first constructs the configuration and then determines the group.

Again, to take up a different problem, let G be a multiply transitive group of degree n whose degree of transitivity is k and let G_1 be the largest subgroup of G which leaves fixed each of a given set of k symbols on which G operates. Let us suppose that G_1 has ν sets of transitivity of degrees $t_1, t_2, \cdots, t_\nu$ where the t's are all different when $\nu>1$ and that it has at least one additional set of transitivity of still another degree. Let s_i be the number of transitive constituents in G_1 each of which is of degree t_i. Then with each set of k symbols from the n on which G operates associate all the remaining symbols in the transitive constituents of degrees $t_1, t_2, \cdots, t_\nu$. We thus have l symbols in the set where $l=k+s_1t_1+s_2t_2+\cdots+s_\nu t_\nu<n$. With each set of k symbols in G form in this way a set of l symbols and let λ denote the number of sets so formed. They constitute a configuration F. Then λ is not greater than the number of combinations of n things taken k at a time.

It is clear that F is invariant under G and that its sets are permuted transitively by G. Moreover, one set of k symbols will appear in just as many of the λ sets of F as any other set of k symbols. Let μ denote this number,

so that each set of k symbols will appear in just μ of the λ sets of F. Then it is easy to show that

$$\lambda = \frac{n(n-1)\cdots(n-k+1)\mu}{l(l-1)\cdots(l-k+1)}, \qquad \mu \leqq \frac{l(l-1)\cdots(l-k+1)}{k!}.$$

Thus, for any given multiply transitive group G whose subgroup G_1 has transitive constituents of different degrees (always realized when $n-k$ is a prime) we have in this way a configuration F left invariant by G, while its sets are permuted transitively by G. (There is nothing to indicate that F may not characterize a larger group containing G as a proper subgroup).

Particular interest attaches to the case when $\mu = 1$, that is, the case in which each set of k symbols appears in only one of the sets which constitute F. When $k \geqq 4$ the conditions thus arising greatly restrict the possible values of n, the restrictions increasing rapidly with increasing k.

Returning to the general case, suppose that a given set M of l elements in F is obtained from each of just ρ sets of k elements, by the method employed in the construction of F. Consider a permutation P of G which transforms M into a set N of F; then the ρ sets each of which leads to M are transformed into ρ sets each of which leads to N; by transforming N to M by P^{-1} we then see that just ρ sets of k each lead to N. Since G permutes the λ sets of F transitively it follows that each of the λ sets in F is obtained from just ρ sets of k elements each. Since each set of k elements occurs in just μ sets of F we then have

$$\lambda = n(n-1)\cdots(n-k+1)/k!\,\rho,$$

while from the previous value of λ we have

$$\rho\mu = l(l-1)\cdots(l-k+1)/k!.$$

11. *Some Generalizations.* By a complete λ-μ-ν-configuration of n elements we shall mean a configuration of n elements taken ν at a time so that each set of μ elements shall occur together in just λ sets. (Compare Netto's *Lehrbuch der Combinatorik,* second edition, p. 325). Then a triple system is a complete 1-2-3-configuration; a quadruple system is a complete 1-3-4-configuration; and so on. A finite two-dimensional geometry $PG(2, p^n)$ is a complete 1-2-(p^n+1)-configuration. In § 8 we have shown the existence of a complete 1-4-5-configuration on 11 elements, a complete 1-5-6-configuration on 12 elements, a complete 1-5-8-configuration on 24 elements, a complete 1-4-7-configuration on 23 elements and a complete 1-3-6-configuration on

22 elements. These examples are sufficient to show the importance of complete λ-μ-ν-configurations for $\lambda = 1$. But little has been done towards a general theory of complete λ-μ-ν-configurations. In the next section we shall treat certain 2-2-k-configurations.

An infinite class of complete 2-3-4-configurations may be constructed in the following manner. Let p be any prime of the form $6m + 1$ and let ρ be a solution of the congruence $t^2 - t + 1 \equiv 0 \bmod p$. The set ∞, 0, 1, ρ is transformed into itself by the group generated by the transformations

$$x' \equiv (x - 1)/x \bmod p \quad \text{and} \quad x' \equiv \rho/x \bmod p,$$

a group whose order is 12. Thence it follows readily that the set ∞, 0, 1, ρ is transformed into $(p + 1)p(p - 1)/12$ quadruples by the linear fractional group modulo p, the order of which is $(p + 1)p(p - 1)$. Since this linear fractional group is triply transitive it follows that each triple of the $p + 1$ elements $\infty, 0, 1, 2, \cdots, p - 1$ occurs among the quadruples in the named set of quadruples, and indeed that each triple occurs the same number of times as any other, whence it follows that each of them occurs just twice. Thence it follows that these quadruples constitute a complete 2-3-4-configuration. In case m is odd (but not when m is even) this configuration breaks up into two equivalent configurations each of which constitutes a complete 1-3-4-configuration, a fact which one may readily verify by showing that the transformations of square determinants in the named linear fractional group then transform ∞, 0, 1 into every triple of the $p + 1$ elements (though as a permutation group it is only doubly transitive).

12. *Certain Complete 2-2-k-configurations.* We shall now treat those complete 2-2-k-configurations of n elements which are formed by n sets of k things each such that each two sets have just two elements in common. Since each of the $\frac{1}{2}n(n - 1)$ pairs of elements occurs just twice and each of the n sets of k elements contains just $\frac{1}{2}k(k - 1)$ pairs it follows that we must have $\frac{1}{2}k(k - 1)n = 2 \cdot \frac{1}{2}n(n - 1)$, whence it is necessary that

$$n = \tfrac{1}{2}k(k - 1) + 1.$$

The case $k = 2$ is entirely trivial. When $k = 3$ we have $n = 4$ and the configuration consists of the four triples which may be formed from four things. When $k = 4$ we have $n = 7$; then it may be shown that the configuration is that which is complementary to $PG(2, 2)$, whence it follows that the group belonging to the configuration is the doubly transitive group of degree 7 and order $7 \cdot 6 \cdot 4$.

When $k=5$ we have $n=11$. It is not difficult to show that there is just one configuration of degree 11 of the type now in consideration and that it is equivalent to the configuration $A_2[11]$ treated in § 9 and that the group belonging to it is doubly transitive and of order $11 \cdot 10 \cdot 6$.

When $k=6$ we have $n=16$. A corresponding configuration may be constructed in the following manner. By means of the adjoining scheme form 16 sets of six letters each by taking for each letter in the scheme the six which are aligned with it (excluding that letter itself). Thus corresponding to A and B we form respectively the sets $BCDEIM$ and $ACDFJN$. The 16 sets formed constitute a complete 2-2-6-configuration of the kind here in consideration, as one may readily verify. The group belonging to the configuration is the doubly transitive group of degree 16 and order $16 \cdot 15 \cdot 12$ 4.

$A \quad B \quad C \quad D$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$E \quad F \quad G \quad H$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$I \quad J \quad K \quad L$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$
$M \quad N \quad O \quad P$
$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

In this case ($k=6$, $n=16$) the configuration is not unique; but the total set of inequivalent configurations seems never to have been determined. (In fact, the general class of configurations treated in this section seems never to have been previously considered). A second configuration for the case $k=6$ and $n=16$ consists of the sets in the following sixteen columns:

A	A	A	B	B	C	B	B	C	C	D	D	A	A	A	F
B	B	C	C	D	D	E	J	E	H	F	G	E	E	I	G
C	D	D	F	E	L	F	K	G	I	M	I	F	G	J	H
H	G	F	G	H	E	N	M	O	K	O	L	K	H	L	J
L	K	J	I	I	M	O	N	J	N	H	N	I	M	M	K
P	O	N	M	J	K	L	P	P	O	P	P	P	N	O	L

To show that this is different from the foregoing 2-2-6-configuration one proves that it belongs to a different group. There exists also a complete 2-2-9-configuration (of the type here studied), consisting of the 37 sets into which the set 1, 7, 9, 10, 12, 16, 26, 33, 34 is transformed by the $37 \cdot 9$ transformations generated by $t' \equiv t+1 \mod 37$ and $t' \equiv 16t \mod 37$.

The configurations which we have named are apparently all the known configurations of the class here in consideration; but there seems to be nothing known to show their non-existence for any value of k greater than unity. In particular, it seems not to be known whether such configurations exist for $k=7$ or 8.

With every configuration of the class here in consideration one may

associate an *adjoint* configuration, in the following manner. Number the sets in the configuration from 1 to n inclusive; let $a_1, a_2, \cdots, a_n$ be the symbols appearing in the configuration. Now form a configuration of the numbers $1, 2, \cdots, n$ by taking for the i-th set the k numbers which designate the k sets in which a_i appears, doing this for $i = 1, 2, \cdots, n$. Then the n numbers appear in n sets of k numbers each. That they form a complete 2-2-k-configuration of the class in consideration is readily shown by observing that if a_i and a_j appear together in the λ-th and μ-th sets of the original configuration, then λ and μ appear together in just two sets of the new configuration, namely, in those determined by means of a_i and a_j. If the second of two configurations is adjoint to the first then the first is also adjoint to the second.